STATIC FAILURE THEORY — DUCTILE MATERIALS

The main reason for looking at the stress is to determine if a part or design will fail. In this chapter we will look at how parts fail under a static or near static load.

Part failure is influenced by the geometry of the part, the loads that are applied to it, and the material from which it is made.

* In general, ductile, isotropic materials in static tensile loading are limited by their shear strength.

Brittle materials are limited by their tensile strengths.

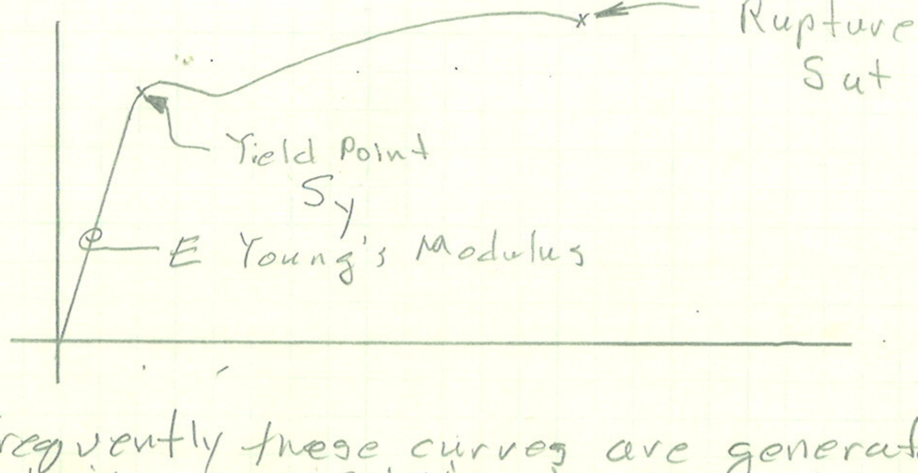
DUCTILE MATERIALS

There are several failure theories but we will only look a the two most reliable methods. $\varepsilon_{max} \geq 5\%$

1) von Mises–Hencky

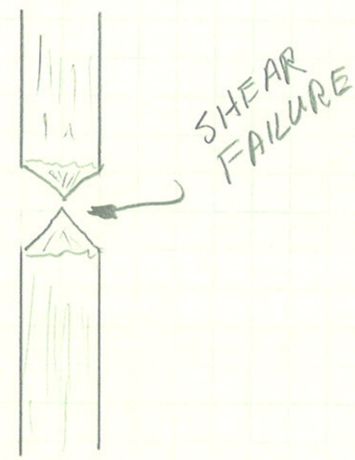
2) Maximum shear-stress

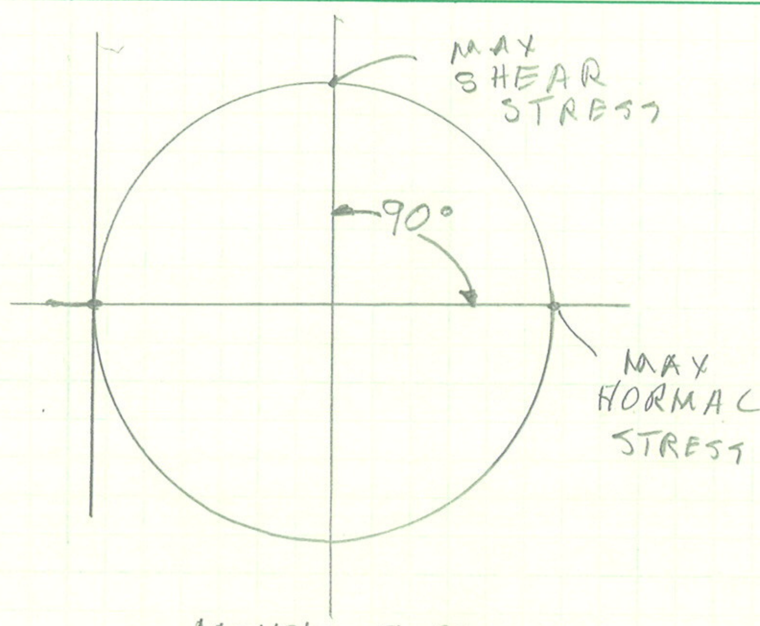
Previously we looked at stress-strain curves



Frequently these curves are generated by placing a specimen in a machine and applying a tensile force until the material fails.

TENSILE TEST

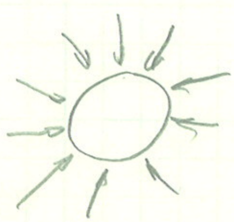




MOHR'S CIRCLE

The degree scale for Mohr's circle is twice what it is in the material, Maximum shear occurs 90° away from the maximum Normal stress but it will be 45° away on the material.

HYDROSTATIC LOADING



Materials have been stressed far beyond their ultimate strength with a hydrostatic loading and they do not fail.

With a hydrostatic loading, there is compressive (Normal) stress but no shear stress.

This leads us to believe that when parts fail the shear plays a strong role. Infact, it relates to the distorsion of the objects which allows the molecules in the structural lattice to slide past one another.

VON MISES STRESS

von Mises effective stress is defined as

"the uniaxial tensile stress that would create the same distortion energy as is created by the actual combined applied stresses".

For the 3D case

$$\sigma' = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3}$$

In terms of applied stresses

$$\sigma' = \sqrt{\frac{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zy}^2)}{2}}$$

For the 2D case let $\sigma_2 = 0$

$$\sigma' = \sqrt{\sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2} \leftarrow 2D$$

or from applied stresses

$$\sigma' = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2} \leftarrow 2D$$

SAFETY FACTOR

If we define the safety factor as:

$$N = \frac{\text{Failure Stress}}{\text{Analyzed Stress}}$$

This N value should always be larger than 1. In fact it is usually significantly larger than 1.

$$1.3 \le N \le 5 \text{ or higher}$$

Everything is known and carefully analyzed

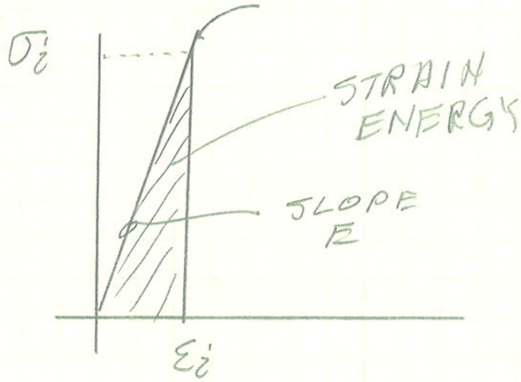
Many things unknown

$$N = \frac{S_y}{\sigma'}$$

S_y — yield stress

σ' — von Mises stress

STRAIN ENERGY



At any point the energy is

$$U = \frac{1}{2}\sigma\varepsilon$$

Extending to 3 dimensions

$$U = \frac{1}{2}\left(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3\right)$$

We know the relationship between stress and strain

$$\varepsilon_1 = \frac{1}{E}\left(\sigma_1 - \nu\sigma_2 - \nu\sigma_3\right)$$

ν = Poissons ratio

$$\varepsilon_2 = \frac{1}{E}\left(\sigma_2 - \nu\sigma_1 - \nu\sigma_3\right)$$

$$\varepsilon_3 = \frac{1}{E}\left(\sigma_3 - \nu\sigma_1 - \nu\sigma_2\right)$$

Substituting these into the equation for strain energy

$$U = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3\right)\right)$$

STRAIN ENERGY

We saw earlier that hydrostatic loading alone did not cause failure. Here we will divide the strain energy developed above into two parts – hydrostatic and distortion energy.

$$U = U_h + U_d$$

Total Energy

hydrostatic energy

distortion energy

The stresses causing hydrostatic energy are the same in all directions so we can divide the principal stresses into:

$$\sigma_1 = \sigma_h + \sigma_{1d}$$
$$\sigma_2 = \sigma_h + \sigma_{2d}$$
$$\sigma_3 = \sigma_h + \sigma_{3d}$$

Adding these together yields

$$\sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_h + (\sigma_{1d} + \sigma_{2d} + \sigma_{3d})$$

or

$$3\sigma_h = \sigma_1 + \sigma_2 + \sigma_3 - \underbrace{(\sigma_{1d} + \sigma_{2d} + \sigma_{3d})}_{0}$$

FOR VOLUMETRIC CHANGE WITH NO DISTORTION THE TERMS IN PARENTHESIS MUST EQUAL ZERO.

SO $\sigma_h = \dfrac{\sigma_1 + \sigma_2 + \sigma_3}{3}$ — average

Now the strain energy U_h can be found. We know that the hydrostatic stress is the same in all directions so using

Total energy → $U = \dfrac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\right)$

we get $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_h$

$$U_h = \frac{1}{2E}\left(\sigma_h^2 + \sigma_h^2 + \sigma_h^2 - 2\nu(\sigma_h\sigma_h + \sigma_h\sigma_h + \sigma_h\sigma_h)\right)$$

$$= \frac{1}{2E}\left[3\sigma_h^2 - 2\nu(3\sigma_h^2)\right)$$

Hydrostatic Energy

$$U_h = \frac{3}{2}\frac{(1-2\nu)}{E}\sigma_h^2$$

Substituting for σ_h

$$U_h = \frac{3}{2}\frac{(1-2\nu)}{E}\left(\underbrace{\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}}_{\text{average}}\right)^2$$

or

$$U_h = \frac{1-2\nu}{6E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\right)$$

TOPS 35500

Using the relationship

$$U_d = U - U_h$$

U_h — hydrostatic energy

U — total energy

U_d — distortion energy

TOTAL

Substituting in

$$U_d = \frac{1}{2E}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2 - 2\nu\left(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3\right)\right) -$$

$$\text{Hydrostatic} \rightarrow \frac{1-2\nu}{6E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2+2\left(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3\right)\right]$$

simplifying

stress

$$U_d = \frac{1+\nu}{3E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_1\sigma_3\right]$$

In General

$$\sigma = \varepsilon E \quad \text{or} \quad \varepsilon = \sigma/E$$

$$\text{Energy} = \frac{1}{2}\sigma\varepsilon \quad \text{or} \quad \text{Energy} = \frac{1}{2E}\sigma^2$$

Using this we set

$$\sigma'^2 = \left[\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_1\sigma_3\right]$$

or

$$\sigma' = \sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_1\sigma_3}$$

— We call this the von Mises stress.

It is based upon the distortion energy and it seems to be better at predicting failure than the other stresses.

For von Mises stresses, the safety factor is computed with

$$N = \frac{S_y}{\sigma'} \quad \begin{matrix} S_y \text{ — yield stress} \\ \sigma' \text{ — von Mises stress} \end{matrix}$$

Another formula we can use that involves applied stresses instead of principal stresses is:

$$\sigma' = \sqrt{\frac{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}{2}}$$

For the 2D case

$$\sigma' = \sqrt{\sigma_1^2 - \sigma_1 \sigma_3 + \sigma_3^2}$$

or

$$\sigma' = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2}$$

~~For Von Mises stress we can compute the safety factor with~~

$$\cancel{N = S_y/\sigma'}$$

~~Yield Stress~~

~~Von Mises Stress~~

EXAMPLE

We have the following stresses in a material.

$$\sigma_x = -500 \quad \sigma_y = 750 \quad \sigma_z = 250$$

$$\tau_{xy} = 100 \quad \tau_{yz} = 250 \quad \tau_{zx} = 1000$$

COMPUTE PRINCIPAL STRESSES

$$\sigma^3 - C_2\sigma^2 - C_1\sigma - C_0 = 0$$

Where

$$C_2 = \sigma_x + \sigma_y + \sigma_z$$

$$C_2 = -500 + 750 + 250 = \underline{500}$$

$$C_1 = \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x$$

$$C_1 = 100^2 + 250^2 + 1000^2 + 500 \cdot 750 - 750 \cdot 250 + 250 \cdot 500$$

$$\underline{C_1 = 1{,}385{,}000}$$

$$C_0 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$$

$$C_0 = (-500)(750)(250) + 2 \cdot (100)(250)(1000) + 500(250) - 750(1000) - 250(100)$$

$$\underline{C_0 = -44{,}400{,}000}$$

$$\sigma^3 - 500\sigma^2 - 1{,}385{,}000\sigma + 44{,}400{,}000$$

$R_1 = 31.7 \quad R_2 = -972 \quad R_3 = 1440$

SO

$\Rightarrow \quad \sigma_1 = 1440 \quad \sigma_2 = 31.7 \quad \sigma_3 = -972$

$$\tau_{max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = \frac{1440 + 972}{2}$$

$\tau_{max} = 1156$ MPa

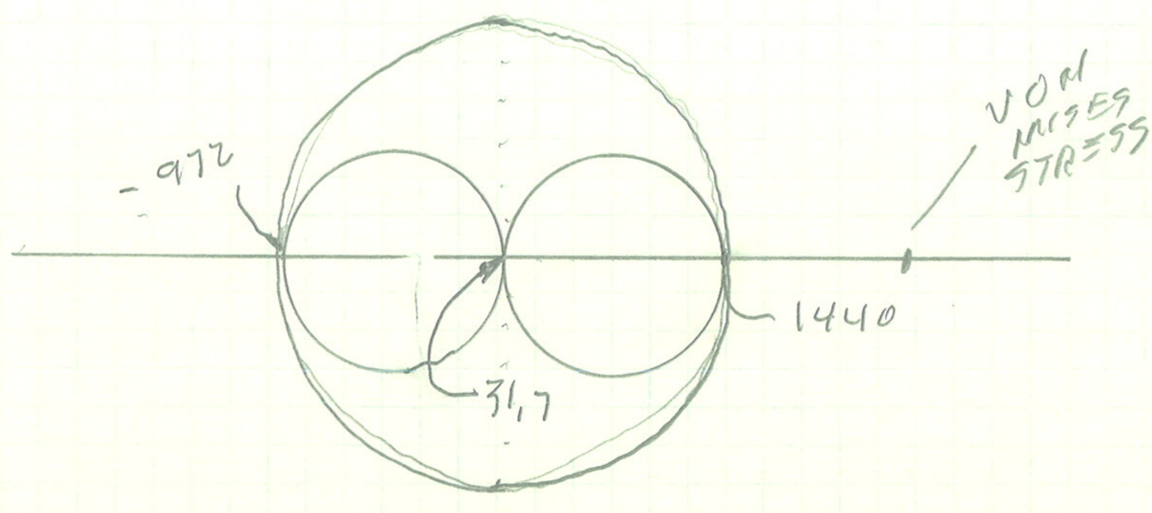
Von Mises Stress

$$\sigma' = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3}$$

$$\sigma' = \sqrt{1440^2 + 31.7^2 + 972^2 - 1440 \cdot 31.7 + 31.7 \cdot 972 + 972 \cdot 1440}$$

$\underline{\sigma' = 2099 \text{ MPa}}$

<u>Mohr's Circle</u>



~~For Applied~~ Stresses

2D STRESSES - EXAMPLE

Where $\sigma_2 = 0$

$$\sigma' = \sqrt{\sigma_1^2 + \cancel{\sigma_2^2} + \sigma_3^2 - \cancel{\sigma_1\sigma_2} - \cancel{\sigma_2\sigma_3} - \sigma_1\sigma_3}$$

$$\sigma' = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3}$$

Let $\sigma_x = 500 \quad \sigma_y = 400 \quad \tau_{xy} = 600$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = \frac{500+400}{2} + \sqrt{\left(\frac{500-400}{2}\right)^2 + 600^2}$$

$$\sigma_1 = \frac{900}{2} + \sqrt{\left(\frac{100}{2}\right)^2 + 600^2}$$

$\sigma_1 = 1052$ MPa

$$\sigma_3 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right) + \tau_{xy}^2}$$

$$\sigma_3 = \frac{900}{2} - \sqrt{\left(\frac{100}{2}\right)^2 + 600^2}$$

$\sigma_3 = -152$

$$\tau_{max} = \frac{|\sigma_1 - \sigma_3|}{2} = \frac{|1052+152|}{2} = 602 \text{ MPa}$$

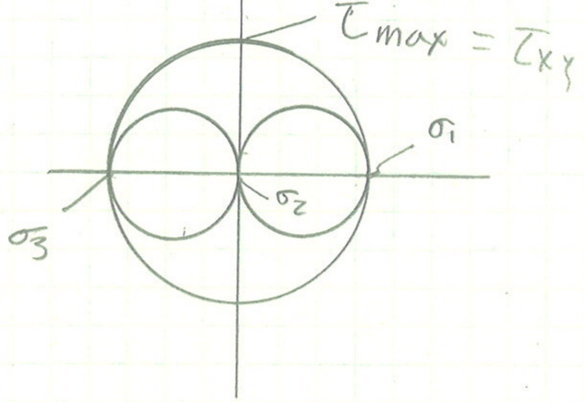
$$\sigma' = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3} = \sqrt{1052^2 + 152^2 + 1052 \cdot 152}$$

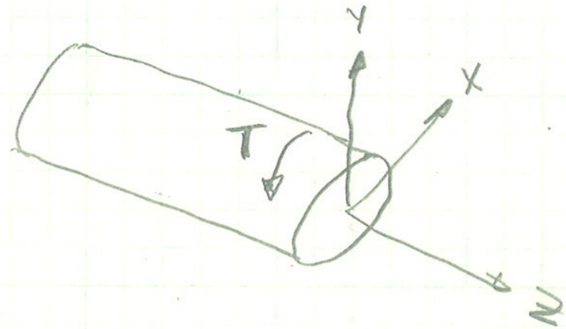
$\sigma = 1135$

TOPS. 35500

PURE SHEAR

We can look at pure shear by looking at a torsional problem





We apply a torque to the cylinder. There are no axial stresses.

Looking at Mohr's circle we see

$$\sigma_1 = \tau_{max} = -\sigma_3$$

$$\tau_{max} = \tau_{xy}$$

$$\sigma_2 = 0$$

From our previous work we know that the von Mises stress can be computed with

$$\sigma' = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3}$$

we compute the safety factor with

$$N = S_y / \sigma'$$

At a safety factor of 1, the von Mises stress is equal to the yield strength

$$\sigma' = S_y$$

substituting we get for $N=1$

$$S_y = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3}$$

For a 2 dimensional case where $\sigma_z = 0$

$$S_y = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_1 \sigma_3}$$

Looking at our pure shear example and substituting in τ_{max} for the σ values

$$S_y = \sqrt{\tau_{max}^2 + \tau_{max}^2 + \tau_{max}^2}$$

$$S_y = \sqrt{3 \tau_{max}^2}$$

or

$$\tau_{max} = \frac{S_y}{\sqrt{3}} = 0.577 S_y$$

The safety factor is 1 so this is where the material fails. This gives us the relationship

$$\boxed{S_{ys} = 0.577 S_y}$$

Which relates the shear yield stress to the axial yield stress.

MAXIMUM SHEAR STRESS THEORY

Before von Mises, researchers looked at tensile test of ductile materials and it was apparent that the failure was due to shear and not tension.

Looking at failure data they derived the the formula

$$S_{ys} = 0.50 S_y$$

Knowing this we can compute the safety factor

$$N = S_{ys} / \tau_{max}$$

OR

$$N = \frac{S_y}{\tau_{max}} = \frac{0.50 S_y}{\tau_{max}} = \frac{\frac{S_y}{2}}{\left(\frac{\sigma_1 - \sigma_3}{2}\right)}$$

$$N = \frac{S_y}{\sigma_1 - \sigma_3}$$

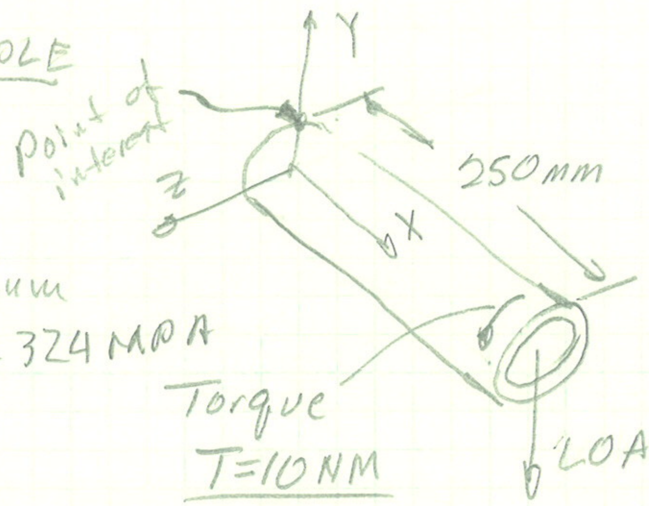
Both the maximum shear stress theory and von Mises stress theory are good for computing the safety factor against failure. The maximum shear stress theory is a little more conservative than von Mises stress

NOTE The maximum Normal stress theory

$$N = \frac{S_y}{\sigma_1} \Leftarrow \text{INVALID}$$

is not valid. Materials can fail eventhough N is greater than 1.

EXAMPLE



Diameter OD = 25mm
Thickness = .5mm

Aluminum
$S_y = 324$ MPA

$T = 10$ NM

SHEAR STRESS DUE TO T

$$\tau_{yz} = T/Q$$

where

meters

$$Q = \frac{\pi(d_o^4 - d_i^4)}{32 r_o} = \frac{\pi(.025^4 - .024^4)}{32(.0125)}$$

$$Q = 4.622 \times 10^{-7}$$

$$\tau_{xz} = 10/4.622 \times 10^{-7} = 21.6 \text{ MPA}$$

STRESS DUE TO LOAD

$$\sigma_x = \frac{Mc}{I}$$ where $c = 12.5$ mm $= .0125$ m

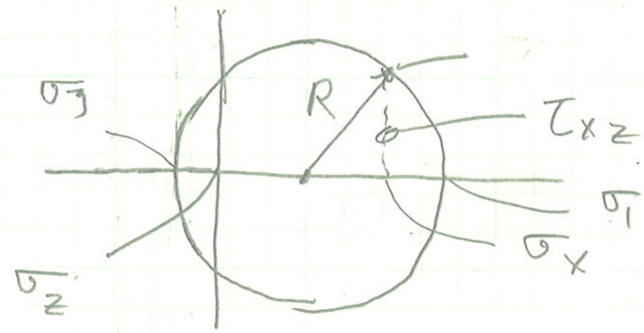
$M = 20 \times .25 = 5$

$$I = \frac{\pi}{64}(d_o^4 - d_i^4) = \frac{\pi}{64}(.025^4 - .024^4)$$

$$I = 2.89 \times 10^{-9}$$

$$\sigma_x = \frac{5(.0125)}{2.89 \times 10^{-9}} \quad \sigma = 21.6 \text{ MPA}$$

Looking at Mohr's circle



$$R = \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right) + \tau_{xz}^2} = \tau_{max}$$

and

$$\sigma_1 = \frac{\sigma_x + \sigma_z}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_z}{2} + \tau_{max} \quad \sigma_3 = \frac{\sigma_x + \sigma_z}{2} - \tau_{max}$$

so

$$\tau_{max} = \sqrt{\left(\frac{21,6}{2}\right)^2 + 21,6^2} = 24,15 \text{ MPa}$$

$$\sigma_1 = \frac{21,6}{2} + 24,15 = 34,95 \text{ MPa}$$

$$\sigma_2 = 0$$

$$\sigma_3 = \frac{21,6}{2} - 24,15 = -13,35$$

von Mises Stress

$$\sigma' = \sqrt{\sigma_1^2 - \sigma_1 \sigma_3 + \sigma_3^2}$$

TOPS 35500

$$\sigma' = \sqrt{34.95^2 - 34.95(-13.35) + (-13.35)^2}$$

$$\sigma' = 43.2 \text{ MPa}$$

Safety factor using maximum distortion theory

$$N = \frac{S_y}{\sigma'} = \frac{324}{43.2} = 7.5$$

Using Maximum Shear theory

$$N = \frac{0.5 S_y}{\tau_{max}} = \frac{(.50)(324)}{24.15} = 6.71$$

TOPS 35500