Addition Theorem for Jacobian Elliptic Functions

Recall the addition formula for sine:

\[ \sin(u + v) = \sin(u) \cos(v) + \sin(v) \cos(u). \]

Since \( \text{sn}(u, k) \) reduces to \( \sin u \) when \( k = 0 \), then we might expect there to be a similar addition formula for \( \text{sn} \), e.g.

\[ \text{sn}(u + v) = ? \]

Consider the rather mundane differential equation

\[
\frac{dv}{du} = -1 \tag{1}
\]

which has the general solution

\[ u + v = A \tag{2} \]

where \( A \) is an arbitrary constant. Now consider the new variables

\[ x \equiv \text{sn}(u, k) \]
\[ y \equiv \text{sn}(v, k) \]

Recalling the derivatives of \( \text{sn} \), we have

\[
\left( \frac{dx}{du} \right)^2 = (1 - x^2)(1 - k^2x^2) \]
\[
\left( \frac{dy}{du} \right)^2 = (1 - y^2)(1 - k^2y^2) \]

and

\[
\frac{d^2x}{du^2} = -(1 + k^2)x + 2k^2x^3 \]
\[
\frac{d^2y}{dy^2} = -(1 + k^2)y + 2k^2y^3 \]

where we have used the fact that

\[ \left( \frac{dv}{du} \right)^2 = (-1)^2 = 1. \]

Using these we have

\[
\frac{d}{du} \left( y \frac{dx}{du} - x \frac{dy}{du} \right) = y \frac{d^2x}{du^2} - x \frac{d^2y}{du^2} \]
\[
= -(1 + k^2)xy + 2k^2x^3y + (1 + k^2)xy - 2k^2y^3x \]
\[
= 2k^2xy(x^2 - y^2). \]
and also
\[
\left( y \frac{dx}{du} - x \frac{dy}{du} \right) \left( y \frac{dx}{du} + x \frac{dy}{du} \right) = y^2 \left( \frac{dx}{du} \right)^2 - x^2 \left( \frac{dy}{du} \right)^2
= y^2(1 - x^2)(1 - k^2x^2) - x^2(1 - y^2)(1 - k^2y^2)
= (y^2 - x^2)(1 - k^2x^2y^2).
\]

Dividing the first by the second, we have
\[
\frac{d}{du} \frac{y \frac{dx}{du} - x \frac{dy}{du}}{y \frac{dx}{du} + x \frac{dy}{du}} = \frac{2k^2xy(x^2 - y^2)}{(y^2 - x^2)(1 - k^2x^2y^2)} = \frac{-2k^2xy}{1 - k^2x^2y^2}.
\]

Since
\[
\frac{d}{du} (1 - k^2x^2y^2) = -2k^2 \left( \frac{dx}{du} + \frac{dy}{du} \right)
\]
we can rewrite the above as
\[
\frac{d}{du} \ln \left( y \frac{dx}{du} - x \frac{dy}{du} \right) = \frac{d}{du} \ln (1 - k^2x^2y^2).
\]

Integrating yields
\[
\frac{y \frac{dx}{du} - x \frac{dy}{du}}{1 - k^2x^2y^2} = B
\]
where \( B \) is an arbitrary constant. Since
\[
\frac{dx}{du} = \text{cn}(u) \text{dn}(u)
\]
and
\[
\frac{dy}{du} = \frac{dv}{du} \frac{d\text{sn}(v)}{dv} = -\text{cn}(v) \text{dn}(v)
\]
we have
\[
\frac{\text{sn}(v) \text{cn}(u) \text{dn}(u) + \text{sn}(u) \text{cn}(v) \text{dn}(v)}{1 - k^2 \text{sn}^2(u) \text{sn}^2(v)} = B. \tag{3}
\]

We now have two integrals of (1), namely (2) and (3), with two arbitrary constants \( A \) and \( B \); however since (1) is only a first order equation \( A \) and \( B \) cannot be functionally independent. \( B \) must be some function of \( A \): \( B = f(A) = f(u + v) \), so
\[
\frac{\text{sn}(v) \text{cn}(u) \text{dn}(u) + \text{sn}(u) \text{cn}(v) \text{dn}(v)}{1 - k^2 \text{sn}^2(u) \text{sn}^2(v)} = f(u + v).
\]
To determine $f$, let’s set $v = 0$. At $v = 0$, $\text{sn}(v) = 0$ and $\text{cn}(v) = \text{dn}(v) = 1$, so

$$\text{sn}(u) = f(u).$$

Therefore

$$\text{sn}(u + v) = \frac{\text{sn}(v) \text{cn}(u) \text{dn}(u) + \text{sn}(u) \text{cn}(v) \text{dn}(v)}{1 - k^2 \text{sn}^2(u) \text{sn}^2(v)}.$$

This is the addition formula for Jacobian elliptic functions, which is a special case of a theorem first proven by Euler concerning functions with algebraic addition formulas. Similar formulas for $\text{cn}$ and $\text{dn}$ can be found by combining this with the relations between the squares of elliptic functions. The proof here, first shown by Darboux, appears in Whittaker and Watson’s *A Course of Modern Analysis* and Akheizer’s *Elements of the Theory of Elliptic Functions*. Other methods of proving the addition theorem also appear in Whittaker and Watson.