Analysis of Functions and Their Graphs as Models of Physical Phenomena: An Integrated Approach Using Maple6

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Academic Setting

This curriculum unit is designed for advanced high school students and will require a full semester of instructional time. I will be employing my unit during the upcoming 2001/2002 school year, while teaching a brand new course of my design at the Career Enrichment Center (CEC) entitled, "Geology of New Mexico." The Career Enrichment Center is a public "magnet"-style high school for students within the Albuquerque Public Schools District (APS), and offers specialized technical courses in physics, electrical/computer engineering, medical-related fields, etc... "Geology of New Mexico" is geared towards advanced college-bound juniors and seniors from many different high schools throughout APS. The course offers two science credits and one upper-level math credit (precalculus/calculus), and therefore will involve instruction in both scientific and mathematical topics. The course will also be similar to an AP style course in that it will be taught at an introductory college level and will carry college credit for students who will attend the University of New Mexico after completing their high school education.

The geological focus of this course is to help students develop a broad background in geology and geologic field methods, including the application of that knowledge to solving many real-life field-based geologic problems. The course is designed around numerous mandatory field excursions designed to help students gain skills and proficiency in the use of critical geologic tools utilized in the field. These tools are used to make critical field observations that can then be used to construct many different types of geologic maps and cross-sections, and collect important data needed to solve many geologic problems. Students will learn about the relationships between geology and fields such as chemistry, physics, biology, and engineering to name a few. Topics to be covered include geophysics, geochemistry, geochronology, mineralogy, petrology, sedimentology and stratigraphy, structural geology, etc... Students will also study geologic hazards and the implications they pose to human existence and survival. This includes the study of volcanology, seismicity, and water-related dangers such as tsunamis and floods.

Unit Prerequisites, Rationale, and Objectives

A minimum of a mathematical background in high school level algebra, geometry, trigonometry, and precalculus/calculus is essential in the quantitative analysis of geologic data and related problems. Mathematics is a scientific field based on logic and reasoning, and has remarkable applications to all areas of science including those mentioned above. One objective of this course is to present students with the mathematical tools needed to make the transition from subjective to quantitative scientific study. Students will learn and apply relevant mathematics in the context of solving actual scientific problems, many of which are directly related to the geosciences. A computer is also an extremely useful tool for solving problems and plotting/analyzing data. Another goal of this course is to address the use of technology in scientific research and problem solving, as well as the role of technology in the classroom. Maple6 (from now on referred to simply as Maple) is a software application designed to manipulate mathematical data both symbolically and graphically. Maple is an integral part of this unit. A minimum of four weeks of this unit will be centered on

Maple and its applications to problem solving and graphing. I will also present much of the visual material (plots, graphs, etc...) of this unit as Maple plots, so that the teachers/students will gain familiarity with diagrams and graphs, created with Maple, at an early stage of this unit.

A thorough mathematical background in both algebra and geometry is assumed of students for the math portion of the course, which will concentrate on trigonometry, precalculus, and some calculus-related concepts and techniques. This includes the study of equations, different classes of functions, and the analysis/graphs of these functions. Function types include linear, nonlinear (quadratics, etc...), polynomial, rational polynomial, trigonometric, and transcendental functions (exponential and logarithmic). Certain functions from each category can be used in various forms as models of physical phenomena. Exponential functions, for example, are used to model both population growth and radioactive decay to a reasonable degree of accuracy. Students often do not develop a thorough understanding regarding the importance of functions, their graphs, and the relation they have to the real world. This unit will be designed to address the importance of functions and the analysis of graphical information with respect to real life models and phenomena. This is the primary goal of this unit. It is imperative that students become aware of the actual uses of mathematics with respect to real life applications. Students are traditionally presented with mathematical content strictly within a purely mathematical context. I feel that part of the reason many students do not respond well to mathematics is due to the fact that they do not understand the relevance of studying mathematics with respect to their lives. My hope is that students will have a much better understanding of real-life mathematical applications upon completion of this unit.

By the end of this semester-long unit, students will have a thorough background in the creation, uses, and analysis of many types of functions associated with real world phenomena. Students will be able to list all the above-mentioned classes of functions, and comprehend the mathematical characteristics associated with each type. They will also be able to give specific examples of function types and actual physical phenomena that can be modeled with each type of equation. Finally, students will be able to apply fundamental analytical techniques towards understanding the evolution and behavior of a given function or model. This includes a graphical analytical approach using Maple6. This unit will give students a firm analytical mathematical background necessary for future studies in pre-calculus, calculus, differential equations, and applied mathematics in general.

Outline of Unit Topics

The topics covered (in order) in this unit are as follows.

1. Instructional review of the definition and basic characteristics of functions, as well as the notation used when working with functions mathematically (~1 week).
2. Review of graphing techniques, creating and shifting graphs of particular functions, and the analysis of graphical information (~2 weeks).
3. The rest of the first nine-week period will involve examining the definitions and characteristics of the types of functions outlined above, along with their respective graphs. This will be accompanied by an in-depth examination of actual mathematical models and equations used to model real world phenomena and problems. Included in this study are models including Newton’s Laws of Motion, gravitation, bacterial growth, radioactive decay, compounded interest, earthquake calculations and the Richter scale, pendulums, sound propagation,

insect flight, catenaries, and more. The teacher may want to choose only a few of these models for an in-depth study rather than attempt to cover too many models at a much shallower level. I have covered a limited amount of these models in the "Implementation" section of this unit. The teacher interested in expanding this base is encouraged to find other examples (some of which can be gotten from the references).

4. This will be followed by a three-week study of basic analytical tools used in mathematics. Students will first learn about rates of change. We will then cover some of the basic concepts and uses of analytic geometry, and solving systems of equations. Another important analytical tool is solving maximum/minimum problems algebraically if no calculus background has been developed. Depending on time and student progress, the teacher may also want to include the study of limits and continuity, differentiation, and integration. I have chosen to exclude these last few topics and leave the teaching of these topics to the discretion of the reader. Most topics will be covered at a fundamental beginners level in this unit, but will provide students with the necessary skills needed to perform analyses of the many functions and models studied throughout the unit thus far, and will provide a firm background for future mathematical study.
5. The next two weeks will then focus on applying analytical methods of analysis to these models and others. Students will learn how to describe physical phenomena mathematically and be able to predict and understand the evolution and behavior of a physical process as governed by a given model. This includes, for example, an understanding of the different ways in which a given process changes with respect to time.
6. The final four weeks of this second nine-week period will be centered on using Maple as a graphical, symbolic, and analytical computing tool. Maple is a software program that specializes in the symbolic manipulation of mathematical statements, and also provides a platform for creating highly sophisticated graphs and plots that can be analyzed and manipulated to extract pertinent information. Students will first learn the basic aspects and commands associated with Maple, and then proceed to learn how to graph functions and data in both two and three dimensions. Students will graph and plot data associated with the functions covered previously in this unit, and analyze the functions graphically. The students will then learn to create computer animations of the time evolution of physical phenomena as described by functions as mathematical models. The unit will then end with the study of composite plots and special plots used with Maple, including implicit plots, and log-scaled plots if time allows.

Context and Background

This part of the unit is given in two sections. The first section deals with general background involving functions and specific applications of functions as models, while the second part is based on Maple6 as an analytical computing tool. It is assumed that most teachers reading this unit are aware of the general mathematics of functions and their graphs. This includes knowledge of the characteristics and manipulation of many function types, as well as the techniques used to analyze these functions. I will therefore highlight certain mathematical information I think necessary to address, while omitting the majority of this assumed background. The first section of this unit will therefore consist primarily of specific examples of phenomena that can be modeled by functions. Many examples are given, but it is left to the reader to find additional examples if a more extensive approach is desired for using this unit in the classroom.

Note: This is not a unit designed to teach students about all aspects of the types of functions involved, but rather to develop a more general understanding of the behavior of a given function both mathematically and graphically and, therefore, the characteristics of each of the major classes of functions. Students will be exposed to the notation used with various types of functions. They will also be shown graphs (ideally using Maple) of the various types of functions, and the ways in which the graphs change with slight modifications of the written function. The goal is to demonstrate, for example, that trigonometric functions such as sine and cosine are oscillatory functions, rather than to present the entire derivation of all trigonometric functions and their associated identities. This should be left to a course on trigonometry.

Functions and Their Graphs

A function is a relationship between quantities (variables) that occurs when the value of one of the quantities can be given uniquely by specified values of the other quantities. Isaac Newton discovered that the acceleration of a body is governed directly by the force applied to it. This is an example of a functional relationship. The variables involved can be either independent or dependent. In an experiment for example, the values of certain variables are fixed while others are allowed to change. The fixed variables are called the independent variables, and the dependent variables are those that change in response to the given value of the independent variable. A function therefore relates dependent variables to independent variables, the only restriction being that each value of the dependent variable is given uniquely by one, and only one, value for each of the independent variables.

Since some functions needn't be evaluated for all possible values of the independent variables, the **domain** of a function is the set of allowed values for the independent variables. The resulting set of values for the dependent variable is called the **range**. The letters *x, y, z* are commonly used to denote variable quantities in mathematics, while letters found in the beginning of the alphabet, i.e. (*a, b, c, d* etc…) are typically used to denote constant quantities. In general, for a function involving only two variables, the letter y is used to denote the dependent variable, and x to denote the independent variable. We therefore write $y = f(x)$ to denote that *y* (the dependent variable) is a function of *x* (the independent variable).

Polynomial and Rational Polynomial Functions

The reader will recall that a polynomial function is given in general as

$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots + px + q$, where *a, b, c, …, q* are constants.

Note that the general form of a polynomial includes linear first-order, quadratic, and higher-order polynomial functions (in the variable x) as special cases. For any polynomial function, at $x = 0, y = f(0) = q$, where *q* is the **y-intercept**, or simply the value of *y* evaluated at $x = 0$. The **roots** of a polynomial equation are found by setting $y = f(x) = 0$, and then finding all values of the variable *x* that make this statement true. The **roots** are also the **x-intercepts** of the graph of the given polynomial function. All polynomial functions have at least one root according to the *fundamental theorem of algebra*, and therefore a polynomial function of degree *n* will have exactly *n* roots (including repeated roots).

Nth degree polynomials given in the general form above are used to model physical phenomena. Volumes are mathematically described by cubic polynomials ($n=3$), while quartic polynomials ($n=4$) are models for light absorption and radiation, fluid flow, and beam deformation.

Algebraic Functions

An algebraic function is one of the form *y = f(x)* and satisfies an equation of the form

$p_0(x)y^n + p_1(x)y^{n-1} + \cdots + p_{n-1}(x)y + p_n(x) = 0$, $p_0(x), \ldots, p_n(x)$ are polynomials in x.

Rational Functions

A **rational function** is given in general as

$f(x) = \frac{g(x)}{h(x)}$, where *g*(x) and *h*(x) are nth degree polynomials.

The domain of a rational function consists of all real numbers that are not roots of the denominator *h(x)*. Recall that in general, a rational function has a vertical asymptote at each number that is a root of the denominator *h(x)*. If *g(x)* and *h(x)* both have a common root, then the graph of the rational function either has a hole (removable singularity) or a vertical asymptote at the common root. Finally, given a rational function whose numerator has degree *n* and whose denominator has degree *k*, if:
$n = k$, then the line $y = a/c$ is a horizontal asymptote.
$n < k$, then the *x-axis* is a horizontal asymptote.

Transcendental Functions

Transcendental functions are best characterized by a set of *elementary transcendental functions*. These functions include exponential, logarithmic, trigonometric, inverse trigonometric, hyperbolic, and inverse hyperbolic functions. I have only examined the certain classes of transcendental functions given above.

Recall that exponential functions are those that can be written in the form $f(x) = x^n$, where *f(x)* is an odd function if *n* is odd, or *f(x)* is an even function if *n* is even.

Exponential functions are characterized by either rapid growth or rapid decay depending on the coefficient of the base and of the power itself. In particular, all exponential growth or decay functions can be written in the form $f(t) = Pe^{kt}$, where *k* is a constant, *f(t)* is the quantity of something at time *t*, *P* is the initial quantity, and *k* is positive for exponential growth and negative for exponential decay.

Logarithmic functions are the inverse functions of exponential functions. Base 10 and base e are the most widely used base values when computing logarithms. The graph of any logarithmic function is produced by the reflection across the line $y = x$ of the corresponding inverse exponential function.

Trigonometric Functions

Recall that trigonometric functions are periodic oscillatory functions and are extremely useful in modeling many types of physical phenomena. Trigonometric functions are typically evaluated at angles given in radians.

Sine and cosine are periodic functions with period 2π, hence for any real number *t*,

$$sin(t) = sin(t \pm 2\pi) \text{ and } cos(t) = cos(t \pm 2\pi).$$

Sine and cosine functions can take on any values between –1 and 1 for any real angle *t*.

Tangent on the other hand is a π -periodic function. The range of the tangent function lies between $-\infty$ and $+\infty$.

Many types of physical phenomena are modeled by functions of the form,

$$f(t) = A\sin(bt+c) \text{ or } g(t) = A\cos(bt+c)$$

If $A \neq 0$ and $b > 0$, then *f(t)* and *g(t)* have amplitude $|A|$, period $2\pi /b$, and phase shift $-c/b$.

Much of the information outlined above is based on works listed in the "references" section of this unit. Please refer to these publications for further review and more extensive coverage of the concepts outlined.

Maple6[2]

Introduction and Overview

The main emphasis of the "Context and Background" section of my unit deals with the characteristics and uses of Maple. It is more or less assumed that most Math teachers are familiar with much of the material presented already, with the exception of the physical applications given in relation to the material above. The last four weeks of this unit are designed as an introductory unit on the uses of Maple as an analytical/graphical software program. The majority of the information presented here is based on the highly informative "Learning Guide" that is included with Maple. In some cases, I have used actual examples given in the Maple learning guide. This curriculum unit covers many critical features of Maple. It does not, however, cover all aspects of Maple, and it is therefore necessary for the reader to become somewhat familiar with Maple prior to teaching this unit.

Many aspects of mathematics are more easily understood if accompanied by geometric or visual-based learning aides. Maple is an extremely useful tool in trying to visualize complicated mathematical information. Maple has all the power of a computing/programming language, but is based on object-oriented "windows-like" interfaces and controls. Maple refers to this as a worksheet-based graphical interface. People with any Microsoft Windows or Mac experience should feel fairly comfortable with Maple’s "point-and-click" interface platform. Students should therefore be manipulating and graphing functions fairly quickly after learning some of the basic Maple interface characteristics. Not only does Maple allow students to graph functions, students are also able to graph functions in two and three dimensions, as well as rotate graphs in all directions in search of different perspectives in which to visualize a function’s graph. Maple is also an extremely powerful symbolic mathematical tool. This type of system is typically referred to as a "Symbolic Computation System" or "Computer Algebra System". Students can use Maple to expand, factor, and simplify algebraic expressions in their symbolic or algebraic form, without having to substitute actual numbers (numerics) into the expression. Maple is therefore able to symbolically obtain exact analytical solutions to many types of problems.

The majority of the information presented here on the use of Maple can be found in many books on Maple, including the learning guide that is included with the Maple software. The greatest resource, however, for learning about Maple and its uses is the on-line help system that is built into the software. Maple can be purchased for educational use as a Student’s Edition, which is considerably less expensive than the cost of the Professional Edition, and can do anything likely to be encountered in a unit of this type.

Maple6 Basics

Maple is an interactive problem-solving environment. Maple is used to produce worksheets that not only contain text, but also contain "live" mathematics, graphs, and animations. Animations are automatically active when you open a Maple worksheet, for example. The Maple worksheet-based graphical interface is straightforward to navigate. Many operations are performed with Maple in the same fashion as with a standard word processor. This includes cutting, pasting, saving files, printing, etc. Maple's graphical interface is similar to Microsoft Word and other word processors in that the top of the screen contains a menu bar, which is underlain by a toolbar containing shortcut buttons for standard operations like cutting and pasting. Below the toolbar is a context bar that changes based on the type of work you are doing. The large area below the context bar is the worksheet environment, or simply your worksheet. This is where all commands, equations, and graphs are displayed. The bottom of the screen contains the status bar, which displays system information as it normally would. Maple is capable of many operations and functions other than those presented here. Most of the aspects of Maple found in this unit are directly related to operations dealing with functions, graphs/plots of functions, and analytical tools used with functional analysis.

Maple is displaying information in standard math notation if (>)? appears in the upper left corner of the worksheet. Maple is displaying Maple notation if no question mark is present. You may toggle between these two formats by going to the Option menu, then Input Display, then Standard Math (Option->Input Display->Standard Display: This shorthand notation will be used to indicate menu selection sequences used in this unit). Maple notation will be used for the rest of this unit. The commands or information that you enter are called the input. All Maple input is entered at the command prompt (>) located somewhere along the left margin of the screen. Bold face type will be used throughout this unit to indicate any text that should be entered (as input) at the command prompt (>). You must end all input with either a colon (:) or semicolon (;). A semi-colon is used when you want Maple to return some sort of output to your input. A colon is used to tell Maple not to respond to your input. Maple responds to your input with some sort of output depending on what you are asking it to do. The output, for example, may consist of a solution set to an equation, or perhaps a graph of an equation. Maple displays all output in the center of the screen directly below your input.

Note: Maple is capable of performing many operations and therefore requires the use of a large number of commands. This part of the unit is by no means a comprehensive list of all possible Maple operations and associated commands. I have listed only those commands and operations relevant to this unit. This primarily includes any Maple commands and operations associated with the analysis of functions and their graphs. Please refer to the learning guide included with the software for additional information regarding other Maple utilities and operations.

Functions with Maple6

Maple is familiar with a wide variety of mathematical functions including sine, natural log, exponential, tangent, and so on (see Mapple fig. 2.2 pg. 42). Maple uses compact notation to represent a given function. Sine is written as sin for example. The function notation is followed by a set of parenthesis containing the argument of the given function, i.e., the expression at which the function should be evaluated. A possible Maple operation may be entered as follows:

>Sin(Pi/4);

$$\frac{1}{2}\sqrt{2}$$

When working with functions in Maple, it is often advantageous to assign a name to a given expression. This will allow you to perform multiple operations with that expression without having to type it every time. The syntax used is **>Name := expression;**

> eqn1 := x*y; (the * symbol is used to denote multiplication)

***eqn1* := xy**

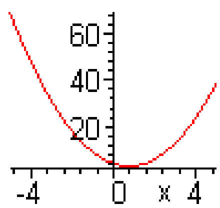
A function can be defined in Maple using arrow notation (→).

>f := x → 2*x^2-3*x+4;

$$f := x \rightarrow 2x^2 - 3x + 4$$

This statement defines **f** as a function that maps the variable **x** into $2x^2-3x+4$. **f** is therefore a function of x and is denoted mathematically as **f(x).** We can now plot the graph of this function without having to enter it again:

>plot (f(x), x=-5..5);



We can also evaluate f(x) at a real number:

>f(2);

6

Please refer to the Maple learning guide for additional operations and commands used with functions.

Graphing with Maple

In order for Maple to run as smoothly and efficiently as possible, Maple only loads the "kernel" at the startup. The "kernel" contains all the fundamental commands and routines necessary for performing basic Maple operations. More advanced operations and commands are loaded into Maple by invoking the proper Maple package. One way to do this is to enter the following syntax:

>with(package name);

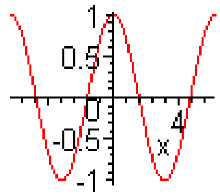
Since this section involves graphing in Maple, we will use the **plots** package. Refer to the Maple learning guide for an extensive list with descriptions of the many Maple packages. We therefore start this Maple section by entering:

>with(plots);

The semicolon tells Maple to respond with a list of all the command names included in the **plots** package.

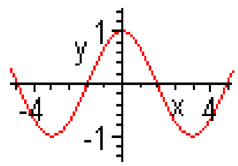
Suppose you want to create a two-dimensional plot of an explicit function, y = f(x). You must enter the function and its domain (interval of definition):

>plot(cos(x), x=-2*Pi..2*Pi);



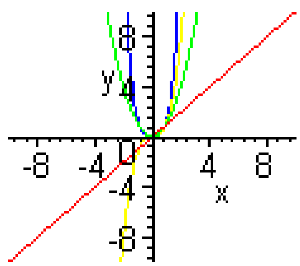
You may also want to limit the range of both the x and y variables.

>plot(cos(x), x=-5..5, y=-1.5..1.5);



To display the graphs of multiple functions in the same plot, enter for example

>plot([x,x^2,x^3,x^4], x=-10..10, y=-10..10);

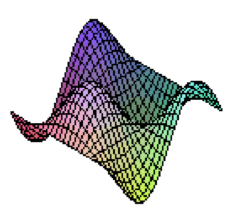


If you plan to cover functions involving three variables, Maple is a great tool for creating three-dimensional graphs. First write the equation in the explicit form (if possible), i.e., $z = f\,(x,y)$. This is an example where z(x,y) = sin(x)cos(y):

>f := (x,y) → sin(x)*cos(y);

$$f := (x,y) \to \sin(x)\cos(x)$$

>plot3d(f(x,y), x=0..2*Pi, y=0..2*Pi);



Maple allows you to manipulate your graphs in a variety of ways. You can actually click your mouse on the graph and rotate it to see what the graph looks like from a variety of different perspectives. You may also change the style and labeling of the coordinate axis. Refer to the Maple learning guide for more information on these and other operations related to graphing.

Animations with Maple6

Maple also allows you to create animation sequences of a given function. You will either use the **animate** or **animate3d** commands depending on the number of variables in your function. The general syntax used is as follows:

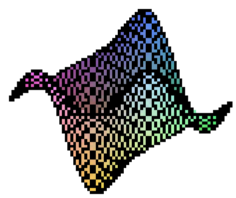
animate(y-expr, x=range, time=range) for two dimensional animations, or

animate3d(z-expr, x=range, y=range, time=range) for three-dimensional animations.

Once you have entered a command of this form, you must then click on the resulting display and then press **Play** from the **Animation** menu. Here is an example of a three dimensional animation. Two-dimensional animations require less commands and are therefore easier to execute once this is understood:

>with(plots):

>animate3d(cos(t*x)*sin(t*y), x=-Pi..Pi, y=-Pi..Pi, t=1..5);



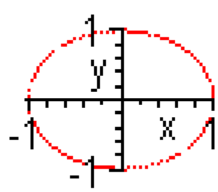
This is obviously a still-frame picture. Try entering the above example to get a better idea of what an actual animation in Maple would look like. Graphs and animations in Maple are a great tool in helping students to visualize the behavior of a function in response to changes in the variables on which it depends.

Special Plots with Maple

Sometimes an equation cannot be written in explicit form. An implicitly defined function is plotted using the **implicitplot** command. Here is an example.

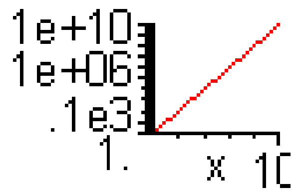
>with(plots):

>implicitplot(x^2+y^2=1, x=-1..1, y=-1..1);



To graph a function ($y=10^x$ for example) using a logarithmic scale on the vertical axis,

>logplot(10^x, x=0..10);



Use the command **semilogplot** for a logarithmic scale on the horizontal axis, and **loglogplot** when you want both axes to have logarithmic scales.

Implementation

The following lessons have been written in a very general way. No anticipated lesson duration times have been given. The time needed to complete a given lesson is highly dependent on the depth of examination, student progress, and the degree to which Maple is utilized within the lesson. Most lessons only require a writing utensil, paper, graph paper, a graphing calculator (TI-83 Plus/TI-86/or TI-89 recommended), and computers loaded with Maple6 (Maple6 was the newest version of Maple at the time of this writing). The following lessons are based on various examples from *Introductory Mathematics Through Science Applications* (Berry et al., 1992). The teacher wishing to use this unit is encouraged to find additional instructive examples pertaining to the mathematical modeling of real-life phenomena.

Note: In the following lessons, the standards are those required by the Albuquerque Public Schools District (APS).

Lesson 1: Loading a Steel Wire: Models Involving Polynomials

Standards/Benchmarks Addressed (see references): 1-5

The following table contains data collected during an experiment designed to study the change in the length of a piece of steel wire in response to weights being attached to it. The wire is 2 meters long before it is stretched and has a diameter of 1 millimeter.

Weight *W* in newtons	Length *l* in meters
20	2.00042
40	2.00086
60	2.00132
80	2.00176
100	2.00219

If we were to plot this data on a graph of weight versus length, we would see that our data points would lie on (or very close to) a straight line. We would therefore conclude that a linear relationship exists between *W* and *l*. Many relationships exist in nature that are linear, but many relationships in nature are nonlinear as well. Bacteria, for example, grow in an exponential fashion, as will be covered later in this section.

We will now use the data given above in an actual example involving Hooke's law. One form of Hooke's law can be written in the following form

$$l = \mu W + l^0$$

The problem is to find values for μ and l^0 based on the data given above. We will then obtain a model for which we can calculate the extension of the wire produced by a weight of 75 newtons. Since $l = \mu W + l^0$ is a linear relationship between l and W, then the graph of this relation will be a straight line with slope μ and y-intercept l^0. If we plot the experimental data as a graph and draw the best fitting line connecting all of the data points, we see that the line crosses the l axis at l = 2.00000 meters. l^0 is therefore given the value 2.00000 meters, which is to be expected since the unextended length of the wire was given as 2 meters. Using the points (0, 2.00000) and (100, 2.00219), we can compute the line's slope as follows:

$$\mu = \frac{2.00219 - 2.00000}{100 - 0} = 0.0000219 mN^{-1}$$

We can therefore rewrite our mathematical model as $l = 0.0000219W + 2.000\,00$

If we evaluate our model by setting W = 75 newtons, l becomes 2.00164 meters. This implies that the wire's length has been extended by 0.00164 meters (1.64 mm) since 2.00164 - 2.00000 = 0.00164. If we take the ration of the extended length to the unextended length, we see that the wire's length has increased by much less than 1% of its original length, i.e., 2.00164/2.00000 = 1.00082.

Interpolation and extrapolation are two very critical concepts in data analysis and the modeling of physical phenomena based on experiment. We now examine these concepts in relation to the linear problem posed above. Plot the data given above on a graph of l versus W, and label each point P^1 , …, P^5 respectively from top to bottom in our data table.

Now suppose that we are interested in the value of l for an imposed load somewhere between 40 and 60 newtons. Intuitively, we conclude that this value of l should lie somewhere between 2.00086 and 2.00132 meters given the data above. Now assuming Hooke's law is valid, we can connect the points P^2 and P^3 with a straight line. We can now read directly from the graph the value of l which corresponds to a given value of W. The method used to obtain information between collected data points in this fashion is called linear interpolation. The term linear is used here to imply that we are using the method of interpolation with a linear relationship.

Notice that the experiment was conducted using weight values of up to 100 newtons. Suppose that we are interested in a certain value of l given by a value of W greater than 100 newtons. We would use the method of linear extrapolation in this case. The method is as follows. By Hooke's law, we can connect P^4 and P^5 with a straight line. We can then extend the line to any value W desired to obtain a corresponding value for l.

It is important to realize that the methods of analysis outlined above give approximations and not precise values. Part of this is due to the fact that the values of *l* measured in an experiment may contain errors of varying degrees. We may however want to use interpolation or extrapolation to actually calculate more precise values of the dependent variable in question. To do so, we use the interpolation/extrapolation formula.

To use the interpolation/extrapolation formula, we must first have an experimentally based data set. Let's call our independent variable *x* and our dependent variable *y*. Let's also assume that we have at least two values for *x*, which we will denote x_1 and x_2, and two for *y*, denoted y_1 and y_2. Now suppose that we are interested in the value of *y*, denoted y_p, that corresponds to some value of *x* between x_1 and x_2, which we will call x_p. Note that the distance $x_p - x_1$ is just a fraction $(x_p - x_1)/(x_2 - x_1)$ of the distance $x_2 - x_1$. We can therefore conclude that the corresponding distance $y_p - y_1$ is the same fraction of $y_2 - y_1$. Setting these two ratio's equal to each other and rearranging gives

$$y_p - y_1 = \frac{x_p - x_1}{x_2 - x_1} \cdot (y_2 - y_1)$$

, which gives

$$y_p = y_1 + \frac{x_p - x_1}{x_2 - x_1} \cdot (y_2 - y_1)$$

This is the interpolation/extrapolation formula, and it works well for both methods. Now let's use this formula to calculate the extended length of the same steel wire as before, but for an attached weight of 30 newtons. We therefore take $x_1 = 20$ and $x_2 = 40$ newtons. We also take $y_1 = 2.00042$ and $y_2 = 2.00086$ meters, with $x_p = 30$ newtons in this case. Thus

$$y_p = 2.00042 + \frac{30-20}{40-20} \cdot (2.00086 - 2.00042) = 2.00064 \text{ meters.}$$

Notice that our answer makes perfect sense since the calculated value lies in between y_1 and y_2. Note also that we used interpolation in this case. To solve a similar problem using extrapolation, we would have chosen x_p to be some weight value greater than those used in the experiment, say 200 newtons for instance.

One final note regarding interpolation and extrapolation: not all relationships are entirely linear. It turns out that the relationship between weight and the corresponding values for extended length of a steel wire is linear for a certain range of weights. As the weight load gets heavier and the wire gets closer to breaking, the graph of this relationship becomes nonlinear and more exponential in nature. This therefore implies that linear extrapolation may involve an associated amount of error as a result of working outside the scope of the data. Linear interpolation is useful for approximating values of the dependent variable for intermediate values of the independent variable. In both cases we are dealing with approximations, hence care should be taken when using either method.

Lesson 2: Bacterial Growth and Models Involving Exponentials

Standards/Benchmarks Addressed: 1-5

Suppose a microbiologist is interested in the rate at which *Escherichia coli* (a type of bacteria) grows under optimal growing conditions. Now lets suppose that the microbiologist measures the number of bacteria cells every ten minutes, and creates the following table based on these observations:

Time Elapsed (min)	0	10	20	30	40	50	60	70	80	90	100
Number of cells	7	10	14	20	27	39	54	76	108	151	213

We can see from the data that the number of cells doubles approximately every 20 minutes. It can be shown that the bacteria actually double in number every 21.3 minutes. Now if we let the number of cells at time *t* be denoted by *N(t),* we can write

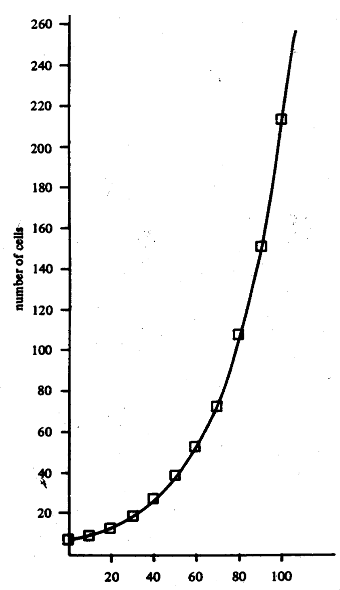
$N(21.3) \cong 2 \times N(0)$

$N(42.6) \cong 2 \times N(21.3) = 2^2 N(0)$

$N(63.9) \cong 2 \times N(42.6) = 2^3 N(0)$*, and in general*

$N(t) \cong N(m \times 21.3) = 2^m N(0).$

We can therefore interpret *m* as the number 21.3 minute intervals in time *t*, i.e., $m=t/21.3$, so we can therefore conclude that $N(t) \cong 2^{t/213} N(0)$. This final formula is used to model the growth of bacteria of the type mentioned earlier. We can therefore deduce the approximate number of cells at any given time *t.* We now want to graph this function, but we will now assume that the initial number of bacteria cells was 7 (see above table) , i.e., $N(0)=7$. Our function can therefore be rewritten as $N(t) \cong 2^{t/213} (7)$. The following diagram has been reprinted from *Introductory Mathematics Through Science Applications* (Berry).



A model of this type is useful since we can deduce the approximate number of bacterial cells at any time *t* using interpolation and extrapolation. Interpolation allows us to approximate the number of cells between observations, while extrapolation helps us determine the number of cells after we stop making observations. It is important to note however, that our model is only an approximation of the actual bacterial growth. This is a result of the fact that our model graphs as a smooth continuous curve, when in reality the graph should show numerous discrete jumps that would represent the division of each individual cell. Even with this approximation, this model still provides a useful description of the growth of bacterial cells. Note also that this model is not unique. We could have written it using any base we want. For reasons that become clear when studying calculus, the base *e* tends to be the most natural base to use when modeling growth and decay processes. We could therefore rewrite the growth equation as

$N(t)=e^{ct} N(0)$, where e = 2.718281828…, and $1/c$ is the time scale required for the number of bacterial cells to increase by the factor *e*, which turns out to be approximately 29.3 minutes.

Now we will examine another mathematical model used to describe the growth of a certain bacterium. Suppose that we are given the formula $M = M_0 e^{0.4t}$, which is used to model the growth of a different type of bacterium. The time *t* in this case is measured in minutes, and M_0 represents the amount of bacteria present at some time $t=0$. Recall that the exponential function e^x can be approximated by the infinite series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

, where we only use the first few terms to approximate the exponential function for small values of x. Now suppose that we are interested in calculating the amount of time required for the bacteria to increase its amount by 20%.

Therefore, if *M* is 20% larger than its initial value M_0, then

$$\frac{M}{M_0} = \frac{120}{100} = 1.2$$

If we assume that the required amount of time is relatively small, we can approximate this function with the infinite series expansion

$$e^{0.4t} + 1 + 0.4t + \frac{1}{2!}(0.4t)^2 + \frac{1}{3!}(0.4t)^3 + \cdots$$

If we further assume that *t* is small enough, we can then use only the first two terms to get

$e^{0.4t} + 1 + 0.4t$, which results in a much simplified model expressed as

$$M = M_0(1 + 0.4t)$$

.

Now since $\frac{M}{M_0}$ =1.2, then this gives 1.2=1+0.4*t*, which upon solving for *t* gives the value 0.5 minutes. If we wanted to insure that we calculated a reasonable value for *t* in this case, we could include the quadratic term in *t* to obtain the expansion

$e^{0.4t} + 1 + 0.4t + \frac{1}{2!}(0.4t)^2$ $=1+0.4t+0.08t^2$.

If we now solve the equation, $1.2=1+0.4t+0.08t^2$ for t, we get $0.08t^2+0.4t-0.2=0$, which is a quadratic equation in t. Using the quadratic formula to obtain a solution for t, we get

$$t = \frac{-0.4 \pm \sqrt{(0.4)^2 - (4)(0.8)(-0.2)}}{(2)(0.08)} = -5.46$$ or 0.46 minutes.

In this case we omit the solution $t = -5.46$ since we are only interested in positive values for t, and therefore a physically realistic solution. Notice that $t = 0.46$ minutes is not very different from our first approximation of $t = 0.50$ minutes. We can therefore conclude that a solution to the equation $1.2 = e^{0.4t}$ is close to $t = 0.46$ minutes. We can continue to approximate the appropriate value for t by adding further terms in the exponential series, but our final value for t is probably adequate for most purposes.

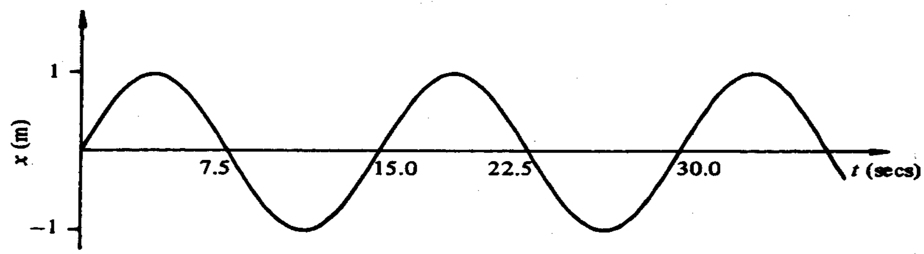
A teacher may also want to also go over realistic examples involving exponential decay processes, such as those used to model radioactive decay of certain atomic elements. Maple is a great tool to use to graph the mathematical models given in these first two examples. Students would then be able to analyze the graphs, and hence the behavior of these models for different values of the independent variable. One may also want to alter the parameter values slightly to see how the graph is affected by those changes.

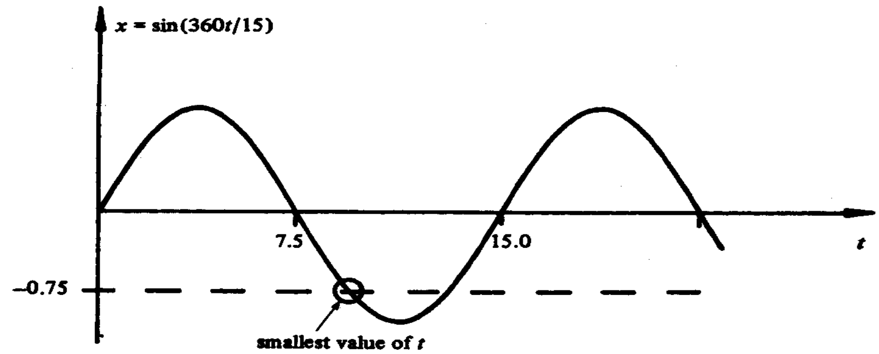
Lesson 3: The Pendulum as Modeled by Trigonometric Functions

Standards/Benchmarks Addressed: 1-5

Suppose we want to model the behavior of a swinging pendulum mathematically. If we assume that there is no drag from the air or mechanical friction involved, a pendulum should continue to swing back and forth indefinitely. If we set a pendulum in motion and then observed its behavior, we would probably notice immediately that its motion is periodic, i.e., the displacement of the pendulum bob from its vertical stable position varies in a periodic way. We may also conclude that the displacement (i.e., position from the bob's central position) of the bob depends on the amount of time that has elapsed since we first set the pendulum in motion, hence the pendulum displacement is a function of time.

We have already examined bacterial growth as a function of time as well, but the exponential function used to model this process is not periodic. It turns out however that trigonometric function are periodic and are therefore very useful in modeling any sort of periodic phenomena. This includes things like planetary motion, water waves, sound waves, electromagnetic waves, alternating electric current (AC current), radio waves, insect flight, and the vibration of the strings on a guitar to name a few. Now examine the figures below.





Above we have two figures reprinted from *Introductory Mathematics Through Science Applications* (Berry 1992). The first figure contains a graph of the horizontal displacement *x* from the bob's central position (i.e., vertical position) as a function of time. The first thing we notice from the graph is that the period of the motion is 15 seconds, and the maximum displacement of the bob is 1 meter.

Now suppose that we want to find a formula describing the displacement *x* as a function of time *t* using a sine or cosine function. We also want to find the smallest value of *t* for which $x = -0.75$. We first notice that the graph in the first figure resembles the graph of the sine function more so than the cosine function. We also note that *x* values in our graph and that of the sine function only vary between ± 1. We can therefore write an initial model as $x = \sin(A(t))$, where *A* represents the angle of the bob measured from the vertical, and is a function of the time elapsed. To find the function $A(t)$, we make the observation that the curve in the first figure has a period of 15 seconds, while the sine curve has a period of 360. We can therefore write the angle function as follows:

$A(t) = 360t/15$, since we want there to be 360 degrees per 15 seconds of time.

This then gives us the formula for *x*, which is $x = \sin(360t/15)$. Notice that after 15 seconds, *A*(15)=360, which means that sin(360)=0 as required. The second figure above displays this relationship. Since the sin function is periodic, it takes on the value – 0.75 many times. We are interested however in the smallest value of *t* that produces this value. If we therefore look at a chart of trigonometric values for sine, we find that the smallest positive angle whose sine is – 0.75 is 228.6 degrees. Substituting this value into the equation for *A* give the relation $228.6 = 360t/15$,and solving for *t*, we obtain

$$\frac{(15)(228.6)}{360} = 9.53$$

seconds. Thus the pendulum reaches a displacement of – 0.75 meters from its central position after 9.53 seconds.

As already mentioned, trigonometric functions are highly useful when modeling wave-like behavior, i.e., physical phenomena that is oscillatory in nature. We now want to consider a function of *x* given by $f(x) = A\sin(kx + c)$. The parameters *A*, *k*, and *c* can be adjusted so that our function behaves like the certain wave-like behavior that we wish to model.

Suppose we throw a rock into a pond of water, hence generating waves on the water's surface. If we set the predisturbed water level equal to zero, then we may want to analyze the amount of vertical displacement (or disturbance) of the water level relative to this predisturbed (zero displacement) level, which depends, let's say, on the size of the rock thrown and the force of impact associated with this rock. We call the size of this displacement (disturbance) the amplitude of the function, which is given by the value of the parameter *A*.

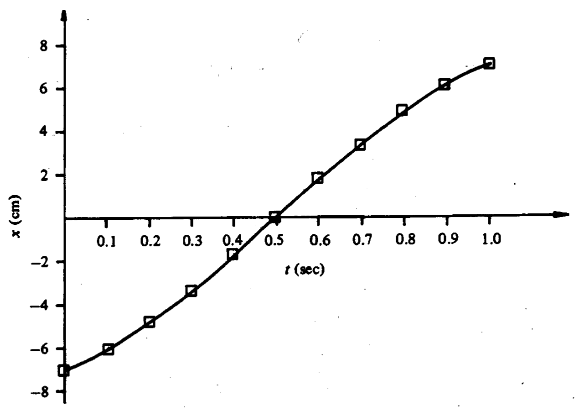
Now let's suppose that we wish to model a certain wave-like phenomena as a function of time. In order to synchronize our function with the time-dependent behavior of the disturbance in question, we must introduce the parameter *c*, which is known as the phase of the function. The period of oscillation of a wave-like disturbance is the amount of time required for the disturbance to oscillate from a given position back to the same position, in other words, for one full wave of the disturbance to take place. Let's say we take the bob of a pendulum and hold it at some angle from the vertical. The position of the bob would represent the displacement or disturbance of the bob from its central vertical position. If we let go of the pendulum and let it swing, the period of the pendulum's behavior would be determined by the amount of time required for the pendulum to make one full swing to and fro and return to the same position from where it was released. It may happen that the oscillatory behavior we wish to model will have the same period as the function we are using as a model, $f(x) = A\sin(kx + c)$ in this case, but that it is slightly out of phase with our model. This means that a full swing of the pendulum (i.e., the period) is the same as our function, but the model we are using might predict that the pendulum will reach its central position at some time *t*, which is different than the actual time measured, say in an experiment. We would therefore have to adjust the parameter *c* in our model to put the model and the actual behavior in phase with each other.

Lets say that we know that the period of one full swing of the pendulum is 2π (the same period of the sine and cosine functions). If the angle *kx+c* changes by 2π , given an initial value of *x*, x_0 , and a final value of *x*, x_1, then since the period of the behavior is 2π , we have that $kx_0 + c + 2\pi = kx_1 + c$. Solving for the corresponding change in *x* gives $kx_1 - kx_0 = 2\pi$, where further manipulation gives $k(x_1 - x_0) = 2\pi$, which finally gives $\Delta x = 2\pi/k$, where Δx represents the change in *x*, which is given as $2\pi/k$. This means that *x* must have changed by $2\pi/k$ for an angle change of 2π radians (measuring angles in radians should be reviewed in class). The quantity $2\pi/k$ is called the period of the function *f(x)*. It is important to note that this period relates to the variable *x* and not to the angle, which has a period of 2π for the sine function. We can therefore fix the value of *k* once we know the period of the wave-like disturbance that we are trying to model mathematically.

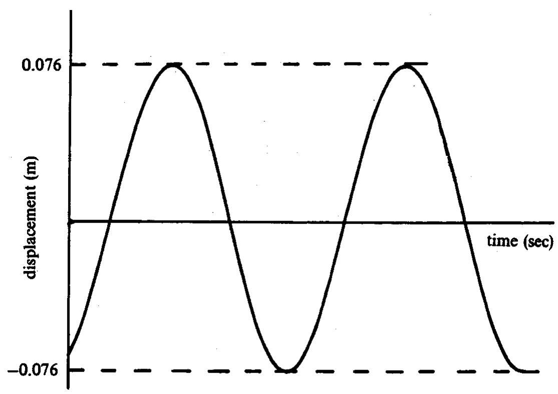
One final note regarding the amplitude *A* of the disturbance. The amplitude of the disturbance represents the overall size of the oscillations being modeled. The amplitude

A is therefore given as half the difference between the largest and smallest values of the disturbance. For the functions sin(*x*) and cos(*x*), the values of these functions for all values of *x* oscillate between – 1 and +1. The amplitude of either function is thus $+1-(-1)/2=1$, and therefore the function $f(x)=A\sin(kx+c)$ has amplitude *A* since its values oscillate between +*A* and – *A*.

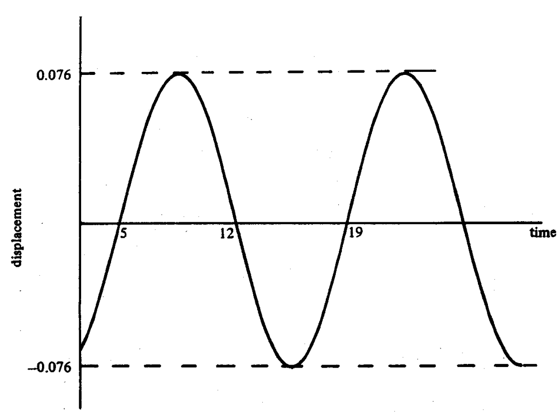
Now let's look at an example. Suppose we perform an experiment involving a pendulum whose motion we wish to model as a function of time. The following three diagrams have been reprinted from *Introductory Mathematics Through Science Applications* (Berry).



Graph of pendulum bob displacement against time over a short interval



Graph of pendulum bob displacement against time over a longer interval.



Notice that the bob's displacement in this case varies between – 0.076 meters to +0.076 meters, hence the amplitude is 0.076 meters. Notice also that the period of the pendulum's motion is 14 seconds. As before, let's try to model the behavior of the pendulum mathematically using the sine function from before, i.e.,

$f(x) = A\sin(kx + c)$. Since the period is $2\pi/k$ and must equal 14 seconds, solving this equality gives that *k* = π /7. Now we need to find the phase angle *c*. For this we can use the first x-intercept, which occurs at *t* = 5. We therefore want the angle $5k + c = 0$, which means that $c = -5k = -5\pi/7$. Now we have all of the information we need to accurately model the motion of this pendulum problem mathematically. Combining all of the known information, the pendulum's motion is described by

$$x = 0.076\sin(\frac{\pi}{7}t - \frac{5\pi}{7}) = 0.076\sin\frac{\pi}{7}(t-5)$$

, where *x* is the displacement of the pendulum from its central position (the vertical) as a function of time.

Assessment

There are many different ways to assess student understanding and comprehension of the material outlined above. I do not believe that tests and quizzes are the only or best way to assess student understanding and progress. Many times students are able to learn (or should I say memorize) the material required for a test in order to get a passing grade, and then forget the material soon thereafter since the material was not truly absorbed mentally. In the real world when someone obtains a job, they are required to use their skills to actually do something of importance, rather than sit down at a table and regurgitate knowledge without actually applying it. I believe that this applies to any form of assessment associated with this unit as well. This unit is motivated by the necessity of presenting mathematical and scientific information in a real world applied sense. I have therefore come up with a few ideas for assessment that are in this spirit.

One possibility would involve a research project in which students find simple or advanced mathematical models used by people in the real world by actually speaking with people from various professions. Bankers, mortgage lenders, and accountants use exponential functions regularly to model compounded interest. Engineers use all the function types covered, and then some, to model a variety of mechanical, structural, chemical, and electrical material properties required to design and engineer a variety of items encountered in everyday life. Many people who work in trades such as plumbing, carpentry, and general contracting, use mathematics as an everyday tool. This could involve simple conversions like inches to meters, or more complicated calculations such as those required to maximize the amount of area that can be contained within a fixed initial length of fencing. Once done, students should present their findings to the class, and then hand in some sort of paper summarizing their research. This type of project would be great for many reasons. One is that students can actually see and hear for themselves that people in the real world actually do use mathematics. The experience may also give them some ideas as to their own future career goals. This type of project is very hands-on and applied, and the kids may have fun with it. Finally, the kids can share their research with the class. Students would hopefully gain an appreciation for the uses of mathematics in society.

Another possibility could involve in-class group projects with the same focus as the examples given above. The teacher could pose a physical problem requiring a mathematical model to describe it. Once the model is constructed, the model could be used to solve many problems. The problem and model could be similar to one of those already posed but with the parameters changed, or it could be entirely new in its scope. The students could then gather in groups and perhaps even have a competition to see which group constructs a correct model the fastest. They may also have to solve a particular problem associated with their model as well. For the students to have an adequate background such that they become capable of model construction, the teacher may want to cover other types of functions and associated models in advance. This should also involve the step-by-step construction of the mathematical model based on known facts, observation, and intuition. This type of in-class project could be given on a regular basis while teaching this unit. One could even do this as each type of function is covered in class.

A final idea involves research similar to the first assessment project, but not necessarily involving people and industry. Students gain valuable research skills when required to find information on their own. They may find information from the library, internet, and in talking with people. I have always been fascinated by the historical evolution of scientific and mathematical thought, and the technology and machinery that has been developed as a result. I love to read about famous (and not so famous) mathematicians and scientists of the past, and the ways in which they discovered or invented the things they did throughout their lives. This is very much associated with mathematical functions as models of physical phenomena. The mathematics used by physicists and engineers to design or create something had to originate from some person who was interested in gaining a better understanding of the physical world surrounding them. It is because of these interests that humanity has evolved both technologically and intellectually to the degree that it has. I therefore believe that students will gain valuable insight as to the motivation for the development of mathematics and the sciences such as physics, when contemplating the ways in which these developments have influenced their lives directly. Students could be required to research something associated with the development and/or history of a certain mathematician or scientist. A solid emphasis for this project would involve the development and evolution of the scientist’s work, and the effects that their work has had during and since their lives. Since this unit is based on models of a mathematical nature, students will want to emphasize the ways in which a person’s work has helped people understand some sort of physical problem or phenomena, and therefore use their understanding to promote the advancement of technology in our society.

Standards Addressed in this unit

1. Strand I: Global Mathematical Process

Content Standard: The student understands and uses mathematical processes.

2. Strand II: Number Sense and Operations

Content Standard: The student demonstrates number sense through experiences with meaningful mathematical problems that focus on number meaning, number relationships, place value concepts, relative effects of operations, and multiple representations to communicate sound mathematical thinking.

3. Strand III: Geometry, Spatial Sense, and Measurement

Content Standard: The student demonstrates an understanding of concepts, properties, and relationships of geometry and measurement through experiences with meaningful mathematical problems that focus on identifying, describing, classifying, visualizing, comparing, estimating, and measuring various aspects of shapes and sizes.

4. Strand IV: Data Analysis, Statistics, and Probability

Content Standard: The students identifies patterns and special features of data and events of chance through experiences with meaningful mathematical problems that focus on comparing, predicting, representing data, and making decisions to communicate mathematical understanding.

5. Strand V: Patterns, Functions, and Algebraic Concepts

Content Standard: The student demonstrates an understanding of algebraic skills and concepts through experiences with meaningful mathematical problems that focus on discovering, describing, modeling, and generalizing patterns and functions, representing and analyzing relationships, and finding and supporting solutions.

References

Much of the information contained in this curriculum unit is based on the first two references listed. Any information regarding the use of Maple6 and its applications to a unit of this type can be found in the Learning Guide that is included with the purchase of the Maple6 software platform. I highly recommend reading a copy of the first reference listed if at all possible. I feel that the material is presented at a very appropriate level for students and teachers likely to be involved in a course/unit of this sort. There is a wealth of examples involving the construction and uses of mathematical models associated with everyday phenomena.

Berry, J., Allan Norcliffe and Stephen Humble. *Introductory Mathematics Through Science Applications*. Cambridge University Press, 1992.

Heal, K.M., M. L. Hansen and K. M. Rickard. *Maple 6 Learning Guide*. Waterloo Maple Inc., 2000.

Advanced Texts and References for Teachers

The following text references are given for the teacher who is interested in additional background material involving mathematical modeling and the applications of mathematical methods to physical problems.

Arfken, George B., and Hans J. Weber. *Mathematical methods for physicists, 5th Ed.* Harcourt Academic Press, 2001.

Bender, Edward A. *An Introduction to Mathematical Modeling.* Mineola, NY: Dover Publications, 1978.

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