

## Money II

I. Last time we derived the equation for compound interest. The equation was

$$F = P(1 + i)^n$$

Where:  $F$  = the future value of money  
 $P$  = the present value of money  
 $i$  = the interest per compounding interval  
 $n$  = the number of compounding periods

### Example

An investor gives you \$150,000 to start your business with the understanding you will give him back 10 times this amount in 5 years. How much must you give the investor in 5 years if the interest rate is 5% per year?

$$F = P(1 + i)^n = 150,000 (1.05)^5 = 150,000 \times 1.2763$$

$$F = \$ 191,442.23$$

This is what the investment would be worth in 5 years at 5% interest. The investor wants 10 times this amount so you will owe him

\$1,914,422 or approximately \$2 million.

II. A Uniform Series of Compound Interest

Define:  $A$  = the end of the period disbursement or receipt from a uniform sum of money.

### Problem

At the end of each year, you are going to deposit a fixed amount of money into an account paying  $i$  percent interest. How much will you have after  $n$  year?

$$F = A(1+i)^{n-1} + \dots + A(1+i)^3 + A(1+i)^2 + A(1+i) + A \quad (2.1)$$

Multiply both sides by  $(1+i)$

$$F(1+i) = A(1+i)^n + A(1+i)^{n-1} + \dots + A(1+i)^3 + A(1+i)^2 + A(1+i)$$

Or

$$F + iF = A(1+i)^n + A(1+i)^{n-1} + \dots + A(1+i)^3 + A(1+i)^2 + A(1+i)$$

Subtracting F from both sides

$$iF = A(1+i)^n + A(1+i)^{n-1} + \dots + A(1+i)^3 + A(1+i)^2 + A(1+i) - F$$

Now substituting equation (2.1) for F on the right hand side of the equation yields.

$$iF = A(1+i)^n + A(1+i)^{n-1} + \dots + A(1+i)^3 + A(1+i)^2 + A(1+i) - A(1+i)^{n-1} - \dots - A(1+i)^3 - A(1+i)^2 - A(1+i) - A$$

Subtracting the terms results in:

$$iF = A(1+i)^n - A$$

or

$$F = A [(1+i)^n - 1]/i \quad (2.2)$$

Example:

You deposit \$500 in the bank at the end of each year for 5 years. How much money will you have after the last deposit if the bank pays 5% interest compounded annually?

$$F = 500(1.05^5 - 1) / 0.05 = \underline{\$2,762.82}$$

F in equation (2.2) is the value of the money at some future time at the end of the investment period. How much is that money worth now? We can compute this by substituting the compound interest formula we derived in the last lecture into equation (2.2). The compound interest formula was:

$$F = P(1+i)^n$$

Substituting yields:

$$F = P(1+i)^n = A [(1+i)^n - 1]/i$$

Solving for A

$$A = P[i(1+i)^n]/[(1+i)^n - 1] \quad (2.3)$$

Example:

You borrow \$200,000 from the bank. How much must you pay monthly to repay the loan in 10 years. The interest is 6% compounded monthly.

$$i = 0.06/12 = 0.005$$

$$n = 10 \text{ years} \times 12 \text{ months/year} = 120$$

$$A = 200,000 [0.005(1.005)^{120}] / [(1.005)^{120} - 1]$$

$$\underline{A = \$2220.41}$$

### Example

Your company needs to buy a lathe for manufacturing parts. The lathe costs \$35,000 but the company will allow you to pay \$400 per month for 10 years. You know you could invest the money at 6 % interest. Which is the best deal?

To solve this, we will compare the present value of the loan to the \$35,000. Both are in present dollars so we can compare them directly. You cannot compare a future value with a present value.

$$A = P[i(1+i)^n] / [(1+i)^n - 1]$$

Solving for P

$$P = A[(1+i)^n - 1] / [i(1+i)^n] \quad (2.4)$$

$$i = 6\% / 12 = 0.005$$

$$A = \$400 \text{ per month}$$

$$n = 10 \times 12 = 120$$

Substituting

$$P = 400[(1.005)^{120} - 1] / [0.005(1.005)^{120}] = \underline{\$36,029.38}$$

So what should you do? The present value of the payment plan is \$36,029.38 and the cost of buying the lathe outright is \$35,000. It appears that buying the lathe outright is the best arrangement.

## Example Problems

### Example I

If you wish to have \$1000 in a saving account at the end of 5 years and the interest rate is 6% paid annually, how much should you put in the saving account now?

*We know from a previous lecture:*

$$F = P(1 + i)^n$$

*Solving this for P results in:*

$$P = F / (1 + i)^n$$

F = the future value = \$1000

i = the interest rate = 6% = 0.06

n = the number of years = 5

$$P = 1000 / (1 + 0.06)^5 = \underline{\$747.26}$$

### Example II

Joe read that a 1 acre parcel of land could be purchased for \$1000 in cash. He decided to save a uniform amount at the end of each month so he would have the required \$1000 at the end of one year. The bank pays 6% interest compounded monthly. How much should Joe save each month?

$$F = A [(1+i)^n - 1]/i \quad (2.2)$$

*Solving this for A results in:*

$$A = iF / [(1+i)^n - 1]$$

$$i = 0.06 / 12 = 0.005$$

$$n = 1 \times 12 = 12$$

$$F = \$1000$$

$$A = (0.005 \times 1000) / [(1.005)^{12} - 1] = \underline{\$81.07}$$

### Example III

On January 1 a man deposits \$5000 in a credit union that pays 8% interest, compounded annually. He wished to withdraw all the money in 5 equal end-of-year sums, beginning December 31<sup>st</sup> of the first year. How much should he withdraw each year?

$$A = P[i(1+i)^n] / [(1+i)^n - 1]$$

$$P = \$5000 \quad n = 5 \quad i = 8\% \quad A = \text{unknown}$$

$$A = 5000 [0.08 (1.08)^5] / (1.08^5 - 1)$$

$$\underline{A = \$1252.28}$$

### Example IV

A \$5000 loan was to be repaid with 8% simple annual interest. A total of \$5800 was paid. How long was the loan outstanding?

$$F = P + Pin$$

*Solving for n*

$$\begin{aligned} n &= (F - P)/Pi && \text{where } i = 8\% && P = \$5000 && F = \$5800 \\ n &= (5800 - 5000)/(5000 \times 0.08) \\ \underline{n = 2 \text{ years}} \end{aligned}$$

### Example V

A sum of money invested at 4% interest, compounded semi-annually, will double in amount in approximately how many years?

$$F = P(1 + i)^n$$

*Solving for n yields*

$$\begin{aligned} F / P &= (1 + i)^n \\ \text{Log } (F/P) &= n \text{Log}(1 + i) \end{aligned}$$

*Or*

$$n = \text{Log } (F/P) / \text{Log}(1 + i)$$

*where*

$$F/P = 2 \quad i = 0.04 / 2 = 0.02$$

$$n = \text{Log } (2) / \text{Log } (1.02) = 35 \text{ periods or } \underline{17.5 \text{ years}}$$