# 2D Triangular Elements

## 4.0 Two Dimensional FEA

Frequently, engineers need to compute the stresses and deformation in relatively thin plates or sheets of material and finite element analysis is ideal for this type of computations. We will look at the development of development of finite element scheme based on triangular elements in this chapter. We will follow basically the same path we used in developing the FEA techniques for trusses.

In both cases, we developed an equation for potential energy and used that equation to develop a stiffness matrix. In the development of the truss equations, we started with Hook's law and developed the equation for potential energy.

$$F = k\Delta x \tag{3.1}$$

$$u = k\int_0^Q x dx = \frac{1}{2}kQ^2 \tag{3.2}$$

From here we developed linear algebraic equations describing the displacement of the nodes (end points) on the truss elements to define a stiffness matrix

$$k = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \tag{3.21}$$

We used this elementary stiffness matrix to create a global stiffness matrix and solve for the nodal displacements using 3.38.

$$KQ = F \tag{3.38}$$

We are going to use a very similar development to create FEA equations for a two dimensional flat plate.

## 4.1 Potential Energy

The potential energy of a truss element (beam) is computed by integrating the force over the displacement of the element as shown in equation 3.2. We will use the same idea but express it in a slightly different manner since we are not working with a one dimensional object such as a beam.

If we apply forces to a thin plate, the plate will deform and in the process store potential energy much the same way a spring will when an external force is applied. If we look at a small element of material in a plate that has been deformed, we can use the stress, $\sigma$ to represent the force in the material and the strain, $\varepsilon$ to represent the displacement of the material. The product of these can be integrated over the volume to compute the potential energy due to external forces applied to the object. This is shown in the equation 4.1.

$$U = \frac{1}{2}\int_V \varepsilon^T \sigma dV \tag{4.1}$$

In 4.1 we are integrating over the entire volume. Since we are studying a flat plate of constant thickness, we can rewrite the equation as

$$U = \frac{1}{2}\int_A \varepsilon^T \sigma t dA \tag{4.2}$$

where:

| | | |
|---|---|---|
| | $\varepsilon$ | is the strain in a differential element of the plate |
| | $\sigma$ | is the stress in a differential element of the plate |
| | t | is the thickness of the plate (we assume it is a constant) |
| | A | is the area of the plate |

In this equation, we are expressing the volume as the area of the plate times the thickness of the plate. We will use this equation for potential energy to develop the stiffness matrix for triangular elements in a thin plate. **Our goal in this development is to replace both the stress and strain terms with linear equations for nodal displacement.**

Equation 4.2 involves both the stress and strain which we do not know. In the following development, we will eliminate both of these terms replacing them with the stiffness matrix and material properties.

## 4.2 FEA Elements

We can take a thin plate and divide it into triangles as shown in Figure 1 below.

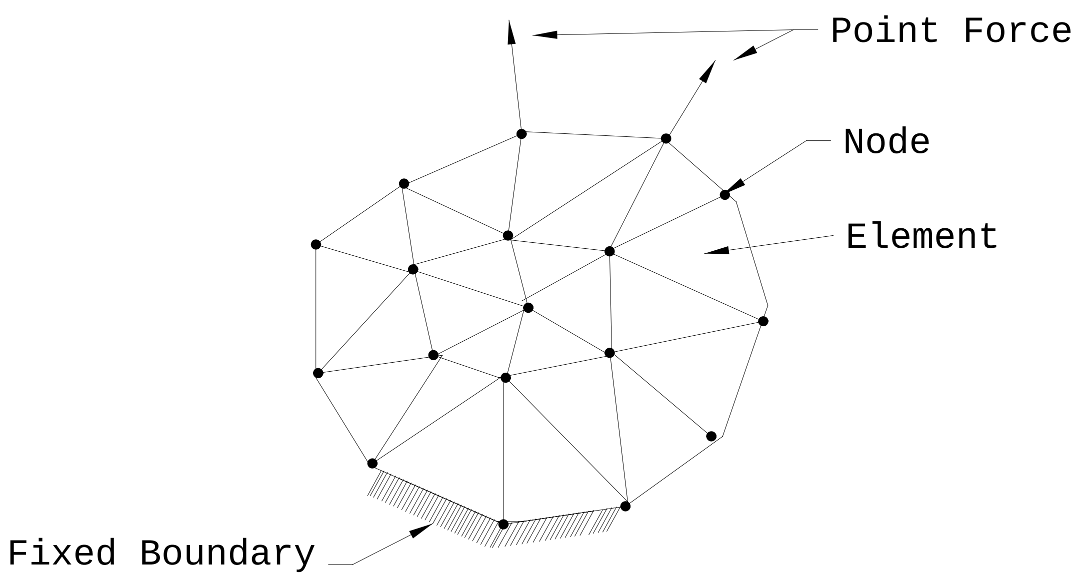



Figure 1 Triangular elements used to approximate a flat plate.

The triangles share vertices with other triangles. The vertices are nodes and triangles are elements. We will use the elements and nodes to approximate the shape of the object and to compute the displacement of points inside the boundary of the object.

The object is fixed along part of the boundary and does not move. External forces are applied at points. These external forces may arise from simple point forces, tractions or forces applied along a length of the boundary, or body forces such as gravity. Regardless of the source, all forces are applied at the nodes only. Tractions, and body forces may be distributed across several nodes but they are still applied at the nodes.

## 4.3 Two dimensional Stress – Strain Relationship

Previously we looked at using finite elements to solve for the nodal displacements along a one dimensional truss member. We derived the equation

$$\sigma = E\varepsilon \tag{3.22}$$

Where $\sigma$ is the stress
$\varepsilon$ is the strain
$E$ is Young's modulus

For the two dimensional case, this becomes a little more complex. If we look at a two dimensional element, we have

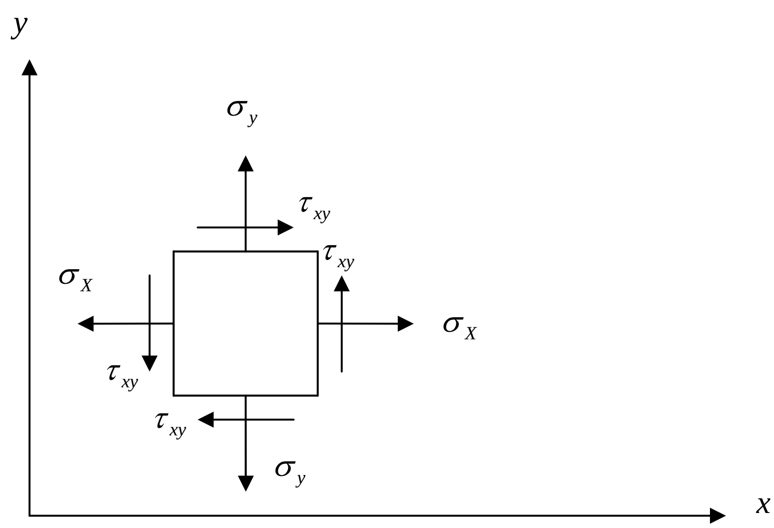



Figure 2 Element showing both normal and shear stresses

The stresses shown in the figure above can be used to write strain equations.

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} \tag{4.3}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} \tag{4.4}$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \tag{4.5}$$

Where:

| | | |
|---|---|---|
| | $\sigma$ | is the axial stress |
| | $\varepsilon$ | is the axial strain |
| | $\tau$ | is the shear stress |
| | $\gamma$ | is the shear strain |
| | $E$ | is Young's modulus |
| | $\nu$ | is Poisson's ratio |

We use the equations above to solve for the stress. First we solve 4.4 for $\sigma_y$ resulting in

$$\sigma_y = E\varepsilon_y + \nu\sigma_x \tag{4.6}$$

Substituting this into equation 4.3 yields

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\left(E\varepsilon_y + \nu\sigma_x\right)}{E} \tag{4.7}$$

or

$$E\varepsilon_x = \sigma_x - \nu E\varepsilon_y - \nu^2\sigma_x \tag{4.8}$$

Solving for $\sigma_x$ gives us

$$\sigma_x = \frac{E}{\left(1-\nu^2\right)}\left(\varepsilon_x + \nu\varepsilon_y\right) \tag{4.9}$$

For the other equations

$$\sigma_y = \frac{E}{\left(1-\nu^2\right)}\left(\varepsilon_y + \nu\varepsilon_x\right) \tag{4.10}$$

and

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \tag{4.11}$$

We can write this in vector form as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{4.12}$$

or

$$\sigma = D\varepsilon \tag{4.13}$$

where

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad \underline{\mathbf{(4.14)}}$$

At this point, we are about half way to developing the stiffness matrix for the triangular mesh. We can use equation 4.13 to rewrite equation 4.2 so that

$$U = \frac{1}{2}\int_A \varepsilon^T \sigma t dA \tag{4.2}$$

becomes

$$U = \frac{1}{2}\int_A \varepsilon^T D \varepsilon t dA \tag{4.15}$$

<u>**We have eliminated the stress term in the equation.**</u> We will go on from here to eliminate the strain term and develop the stiffness matrix.

## 4.4 2D Triangular Elements

In the two dimensional truss problem, we computed the displacements of the nodes and we will do the same here. We will have displacements in the X and Y directions and we will number them as shown in Figure 3.

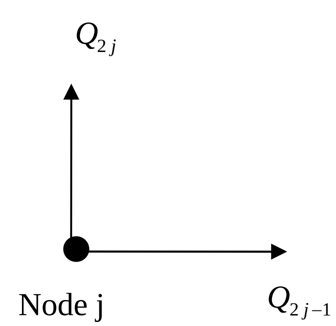



Figure 3 Diagram showing the numbering of nodal displacements.

For a single triangle we have

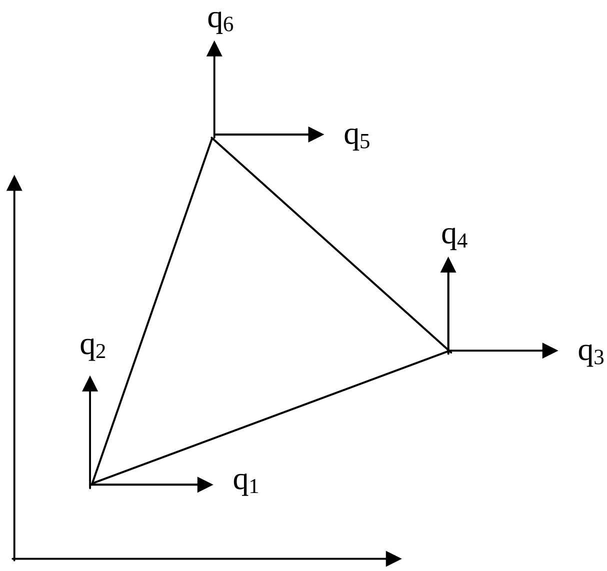



Figure 4  Diagram of a triangle showing the numbering of the displacements of its nodes.

We can write the local displacement vectors for each triangle as

$$q = \{q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5 \quad q_6\}^T \tag{4.16}$$

For the whole object the global vectors can be written as

$$Q = \{Q_1 \quad Q_2 \quad Q_3 \ldots Q_n\}^T \tag{4.17}$$

Which includes all of the $q_n$ terms.

## 4.5 Shape Functions

We are going to compute the displacement of the nodes at the triangle vertices but we also need to compute the displacement for points inside the triangle. We will use shape functions to interpolate the nodal displacements to compute the displacements of arbitrary points inside the triangles.

We will start by moving only one point on the triangle and holding the other two fixed. We can draw both the deformed and non-deformed triangles on top of one another as shown in Figure 5.

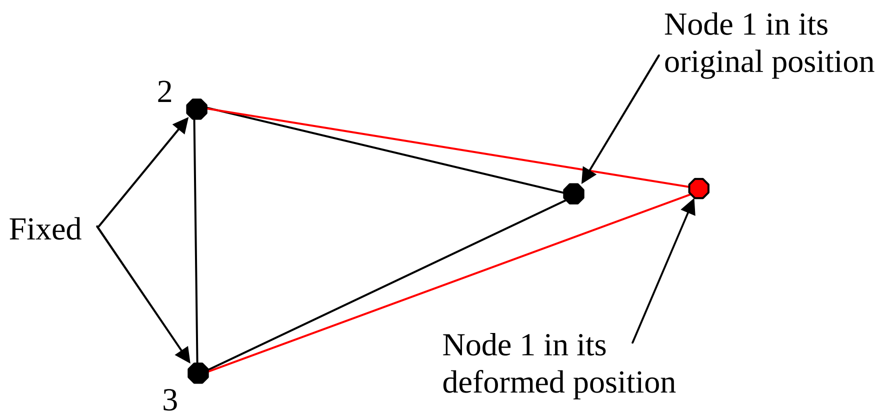



Figure 5  Triangle in both non-deformed and deformed states.

From the diagram above, it is easy to see that points near nodes 2 and 3 will not move as far as points near node 1 when the triangle deforms.  We assume the deformation is linear and we can compute the displacement inside the triangle using an interpolation technique based on areas.  The area of a triangle is

$$Area = \frac{1}{2} Base \times Height \tag{4.18}$$

We are holding two points fixed and moving the third, so the base of the triangle base remains fixed and only the height is changing.  This makes the change in area a linear function with its only variable being the change in height of the triangle or the displacement of the node.

All three nodes of the triangle can be displaced and we will write three linear functions to describe the displacement of an interior point due to the displacement of each of the triangles points.  The displacement of the interior node will be computed by summing the displacement due to each three triangle nodes.

The interior point in Figure 6 divides the triangle into 3 regions.

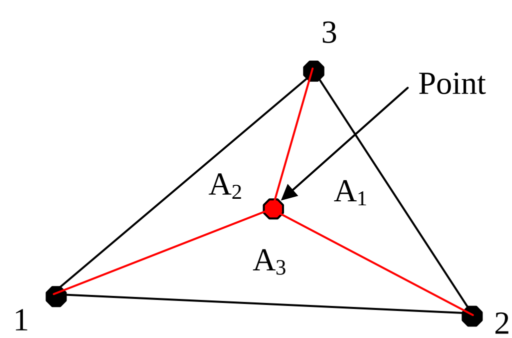



Figure 6  Interior point divides a triangle into 3 regions.

All 3 nodal points may move and the motion of the interior point is some combination of their displacement.  Let $A_1$, $A_2$, and $A_3$ be the areas of each of triangular regions and A the total area of the element. We can see from the diagram that the area of the triangle is equal to the sum of $A_1$, $A_2$, and $A_3$. This is shown in Equation 4.19.

$$A = A_1 + A_2 + A_3 \tag{4.19}$$

We can derive shape functions

$$N_1 = \frac{A_1}{A}, \; N_2 = \frac{A_2}{A}, \text{ and } N_3 = \frac{A_3}{A} \tag{4.20}$$

The displacement of the interior point can be computed with the equations 4.21 and 4.22. The displacement u is in the X direction and v is in the Y direction.

$$u = N_1 q_1 + N_2 q_3 + N_3 q_5 \tag{4.21}$$
$$v = N_1 q_2 + N_2 q_4 + N_3 q_6 \tag{4.22}$$

The shape functions are not independent of one another because:

$$N_1 + N_2 + N_3 = 1 \tag{4.23}$$

Knowing two of the shape functions makes it possible to compute the third. Because of this we can let

$$N_1 = \xi, \; N_2 = \eta, \text{ and } N_3 = 1 - \xi - \eta \tag{4.24}$$

Substituting these equations into 4.21 and 4.22 yields

$$u = (q_1 - q_5)\xi + (q_3 - q_5)\eta + q_5 \tag{4.25}$$
$$v = (q_2 - q_6)\xi + (q_4 - q_6)\eta + q_6 \tag{4.26}$$

We can use these same shape functions to compute the coordinates of a point interior to the triangle where $x_1$, $y_1$, $x_2$, $y_2$, and $x_3$, $y_3$ are the coordinates of the triangle's vertices and x and y are the coordinates of an arbitrary point inside the triangle.

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 \tag{4.27}$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 \tag{4.28}$$

Making the substitution to $\xi$ and $\eta$ gives us

$$x = (x_1 - x_3)\xi + (x_2 - x_3)\eta + x_3 \tag{4.29}$$

and

$$y = (y_1 - y_3)\xi + (y_2 - y_3)\eta + y_3 \tag{4.30}$$

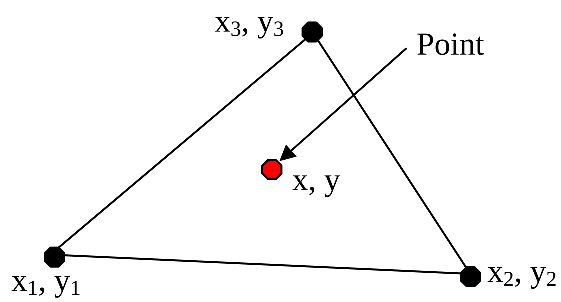



Figure 7 Using the coordinates of a point in the triangle to compute the shape functions for the point.

These equations can be used to compute the shape functions. Given some point in the triangle (See Figure 7). We know $x_1$, $y_1$, $x_2$, $y_2$, $x_3$, $y_3$, and x, y so we solve equations 4.29 and 4.30 for $\xi$ and $\eta$. If we know the displacements at the nodes, we can use the same shape functions to compute the displacement for the point at x, y.

## 4.6 Elementary Solid Mechanics

If we have a small element of material

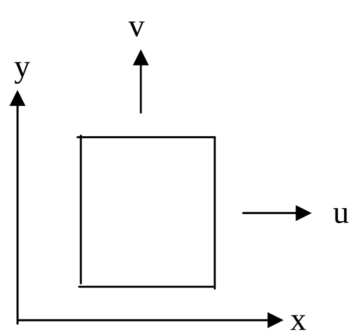



Figure 8 Displacement in a small element of material.

And u and v are the displacements across the element, then we can write the strain as

$$\varepsilon = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \end{Bmatrix} \begin{array}{l} \leftarrow \text{Strain in the X direction} \\ \leftarrow \text{Strain in the Y direction} \\ \leftarrow \text{Shear strain} \end{array} \tag{4.31}$$

We have equations for u and v but these equations are expressed in terms of $\xi$ and $\eta$ not x and y. But, using the chain rule

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi} \tag{4.32}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta} \tag{4.33}$$

We can write this in matrix form as

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} \tag{4.34}$$

or

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{4.35}$$

We can use equations 4.29 and 4.30 to compute the derivatives in the matrix above.

$$\frac{\partial x}{\partial \xi} = x_1 - x_3 \tag{4.36}$$

$$\frac{\partial x}{\partial \eta} = x_2 - x_3 \tag{4.37}$$

$$\frac{\partial y}{\partial \xi} = y_1 - y_3 \tag{4.38}$$

$$\frac{\partial y}{\partial \xi} = y_2 - y_3 \tag{4.39}$$

We can simplify the equations somewhat by letting

$$x_{ij} = x_i - x_j \tag{4.40}$$

and

$$y_{ij} = y_i - y_j \tag{4.41}$$

Substituting equations 4.36 through 4.41 into equation 4.35 yields

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} \tag{4.42}$$

From linear algebra we know that if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{4.43}$$

Then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{4.44}$$

We also know that the Jacobian of a matrix is defined as

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{4.45}$$

and incidentally, the area of the triangle can be defined as

$$Area = \frac{1}{2} \left| \det J \right| \tag{4.46}$$

Putting this together gives us

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det J} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} \tag{4.47}$$

Or by multiplying through

$$\frac{\partial u}{\partial x} = \frac{1}{\det J} \left( y_{23} \frac{\partial u}{\partial \xi} - y_{13} \frac{\partial u}{\partial \eta} \right) \tag{4.48}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\det J} \left( -x_{23} \frac{\partial u}{\partial \xi} + x_{13} \frac{\partial u}{\partial \eta} \right) \tag{4.49}$$

We can now use equation 4.25 to compute the remaining derivatives on the right hand side of equation 4.47

$$u = (q_1 - q_5)\xi + (q_3 - q_5)\eta + q_5 \tag{4.25}$$

so

$$\frac{\partial u}{\partial x} = \frac{1}{\det J}\left(y_{23}(q_1 - q_5) - y_{13}(q_3 - q_5)\right) \tag{4.50}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\det J}\left(-x_{23}(q_1 - q_5) + x_{13}(q_3 - q_5)\right) \tag{4.51}$$

Using a similar process for v we find that

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det J}\begin{Bmatrix} y_{23}\frac{\partial v}{\partial \xi} - y_{13}\frac{\partial v}{\partial \eta} \\ -x_{23}\frac{\partial v}{\partial \xi} + x_{13}\frac{\partial v}{\partial \eta} \end{Bmatrix} \tag{4.52}$$

From equation 4.26 we can again compute the derivatives on the right hand side of equation 4.52

$$v = (q_2 - q_6)\xi + (q_4 - q_6)\eta + q_6 \tag{4.26}$$

Resulting in

$$\frac{\partial v}{\partial x} = \frac{1}{\det J}\left(y_{23}(q_2 - q_6) - y_{13}(q_4 - q_6)\right) \tag{4.53}$$

$$\frac{\partial v}{\partial y} = \frac{1}{\det J}\left(-x_{23}(q_2 - q_6) + x_{13}(q_4 - q_6)\right) \tag{4.54}$$

So the strain defined in equation 4.31

$$\varepsilon = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \tag{4.31}$$

Becomes

$$\varepsilon = \frac{1}{\det J}\begin{Bmatrix} y_{23}(q_1 - q_5) - y_{13}(q_3 - q_5) \\ -x_{23}(q_2 - q_6) + x_{13}(q_4 - q_6) \\ -x_{23}(q_1 - q_5) + x_{13}(q_3 - q_5) + y_{23}(q_2 - q_6) - y_{13}(q_4 - q_6) \end{Bmatrix} \tag{4.55}$$

We can simplify this equation by combining terms. There are many relationships we can make using the x and y terms by realizing that

$$y_{12} = y_1 - y_2 \tag{4.56}$$

Now adding and subtracting $y_3$ from the right hand side

$$y_{12} = y_1 - y_3 - y_2 + y_3 \tag{4.57}$$

so

$$y_{12} = y_{13} - y_{23} \tag{4.58}$$

Using this type of substitution allows us to rewrite equation 4.55 as

$$\varepsilon = \frac{1}{\det J}\begin{Bmatrix} y_{23}q_1 + y_{31}q_3 + y_{12}q_5 \\ x_{23}q_2 + x_{13}q_4 + x_{21}q_6 \\ x_{23}q_1 + y_{23}q_2 + x_{13}q_3 + y_{13}q_4 + x_{21}q_5 + y_{12}q_6 \end{Bmatrix} \tag{4.59}$$

Writing the equation in matrix form we get

$$\varepsilon = Bq \tag{4.60}$$

Where B is a 3x6 element strain displacement matrix relating 3 strains to 6 nodal displacements.

$$B = \frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \tag{4.61}$$

Substituting Equation 4.60 into our potential energy equation 4.14

$$U = \frac{1}{2}\int_A \varepsilon^T D\varepsilon t dA \tag{4.15}$$

gives us

$$U = \frac{1}{2}\int_A q^T B^T DBqtdA \tag{4.62}$$

If we are looking at a single triangle we can rewrite this equation as

$$U_e = \frac{1}{2}\int_e q^T B^T DBqt_e dA \tag{4.63}$$

The thickness of the plate t is a constant as are the matrices B and D. We can move these outside the integral resulting in

$$U_e = \frac{1}{2} q^T B^T DBt_e \left(\int_e dA\right) q \tag{4.64}$$

We recognize the integral $\int_e dA$ as just the area of the triangle so our equation becomes

$$U_e = \frac{1}{2} q^T t_e A_e B^T DBq \tag{4.65}$$

We can now represent the stiff matrix for the triangle as

$$k_e = t_e A_e B^T DB \tag{4.66}$$

With this substitution, our potential energy U becomes

$$U_e = \frac{1}{2} q^T k_e q \tag{4.67}$$

We can sum these individual triangles to compute the strain energy over the entire plate giving us the equation

$$U = \sum_e \frac{1}{2} q^T k_e q \tag{4.68}$$

or

$$U = \frac{1}{2} Q^T KQ \tag{4.69}$$

In this equation Q is the global displacement vector which is the sum of all the local displacement vectors and K is the global stiffness matrix which is the sum of all the local stiffness matrices.

We now have what we need to solve for the displacements in our familiar equation

$$KQ = F \tag{3.38}$$

## 4.7 Computing Displacements

The displacements in a two dimensional plate can be computed using a technique similar to the one used for trusses. There are several steps involved in this computation which is outlined below. This type of computation is very systematic and makes an excellent candidate for a computer program.

1. The first step is called meshing. It divides the plate being studied into a non-overlapping mesh of triangles. These triangles should be roughly the same size and should be as close to equilateral as possible. They must also share vertices so that all triangles that share a side also share vertices. This process can be done by hand but most modern finite element software will create this mesh automatically.

   The triangles and their vertices are numbered. The triangles are called elements and the vertices nodes. They correspond to the elements and nodes in the finite element truss problem we examined in the previous chapter.

2. The next step is to compute the stiffness matrix for each triangle. It is computed using the equation

$$k_e = t_e A_e B^T D B \tag{4.66}$$

where: $t_e =$ thickness of the triangle.
$A_e =$ area of the triangle

$$B = \frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \tag{4.61}$$

$$\det J = x_{13} y_{23} - y_{13} x_{23} \tag{4.70}$$

$$Area = A_e = \frac{1}{2}\left|\det J\right| \tag{4.46}$$

$$D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{4.14}$$

The terms $x_{12}$ and $y_{12}$ can be computed from the coordinates of the vertices with the relationships

$$x_{12} = x_1 - x_2 \tag{4.71}$$

and

$$y_{12} = y_1 - y_2 \tag{4.72}$$

3. The stiffness matrix $k_e$ for each triangle is summed in the global stiffness matrix K. The degrees of freedom (DOFs) of the nodes are used to determine which row and column of the global stiffness matrix to use when summing the local stiffness matrix. This global stiffness matrix is symmetric.

4. The global stiffness matrix is then used to write equation 3.38,

$$KQ = F \tag{3.38}$$

   where K is the global stiffness matrix, Q is the displacement vector we want to compute, and F is a vector defining the external forces applied at each node. This is the same type of problem we solved for computing the nodal displacement in trusses.

5. Our next step before actually solving the problem is to apply the constraints used to fix the plate in space. We apply these constraints by removing the rows and columns in the problem above which are associated with the fixed DOF. This reduces the size of the problem and changes it from a singular problem which has no solution to one we can solve.

6. The final step is to solve the problem for the displacement of the triangular nodes. The problem can be solved with Gaussian elimination or other techniques.

## 4.8 Computing Stresses

At this point we have computed the nodal displacements of the triangles but we have not computed the stresses in the plate. Looking back through the development we have the equation for stress

$$\sigma = D\varepsilon \tag{4.13}$$

Where

$$D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{4.14}$$

We also know that

$$\varepsilon = Bq \tag{4.60}$$

We can put these equations together to create an equation we can use to computed the stresses. It is

$$\sigma = DBq \tag{4.73}$$

This equation can be expanded to create

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \tag{4.74}$$

which we will solve for the stresses. We use the displacements for the nodes of a triangle to compute the stresses for the triangle. The odd numbered displacements are the horizontal displacements and the even numbered displacements are the vertical displacements. There are six displacements in the equation because there are 6 degrees of freedom for each triangle.

The question is, where is this stress? It cannot be placed at one of the nodes because all of the nodal displacements are used to compute the stress. The stress we have computed is actually the stress over the entire triangle. Remember that we assumed that the displacements in the triangle varied linearly across the triangle and we came up with a linear interpolation function to compute this displacement. We know from equation 4.13 that the stress is a function of the strain and from equation 4.31 we see that the strain

$$\varepsilon = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \tag{4.31}$$

is a function of the derivative of the displacements. The shape function we are using interpolates the displacements with linear functions so the derivative strain must be a constant for each triangle. We call this a first order or constant strain mesh.

If the strain is a constant, the stress is also a constant and if the stress is a constant for each triangle, there must be a stress discontinuity at the triangle boundaries.

Well, we know from a physical stand point this is not the case. The stress is a smooth continuous function across the triangle boundaries. So what do we do?

Here we fudge a little. We assume the stress we have computed is near the center of gravity of the triangle and that it does transition smoothly to the center of the next triangle. This is not exactly what we have computed but it is a reasonable approximation.

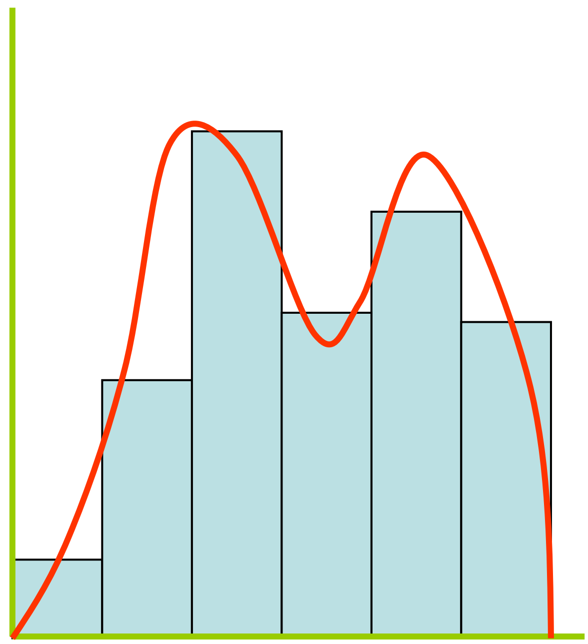


**Figure 9 - A curve approximated with rectangles.**

The problem we are facing is similar to approximating the area under a curve with rectangles as shown in Figure 9. We know from calculus that we can improve the approximation by decreasing the size of the rectangles. The same holds true for computing the stresses in the plate. If we decrease the size of the triangles thus increasing their number, the accuracy of our computations will improve. There is a trade off.  Increasing the number of triangles significantly increases the time to solve the problem but is reduces the error involved in the computations.

Another way to improve the accuracy of the computations is to improve the way we are interpolating the displacements across the triangles.  In our current development we used a linear function to interpolate this displacement but we could use a higher order polynomial.  This is exactly what Mechanica does.  Instead of creating more triangles, it increases the order of the interpolating polynomial.  If the polynomial order, used as an interpolating function has order quadratic or above the stress will not be a constant across the triangle.

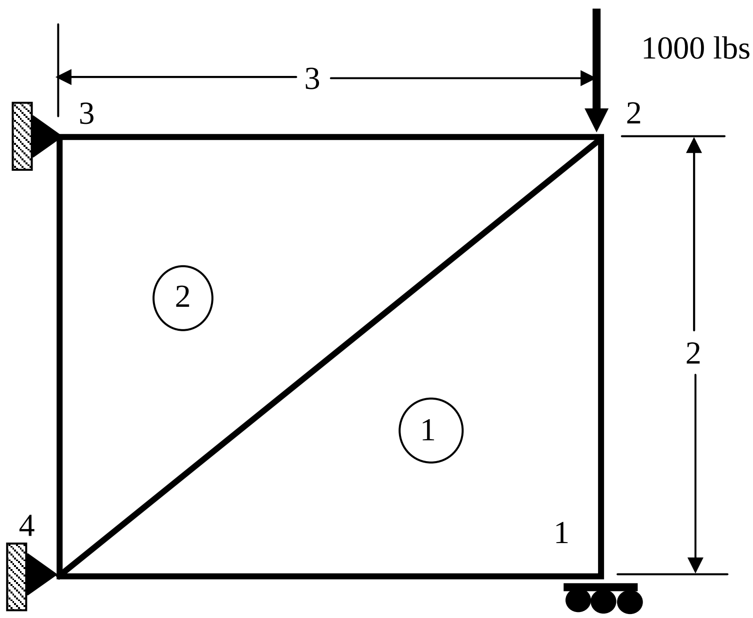
## 4.9 Example

Consider the small plate shown below. The plate is welded in place on the left side to a very stiff support and rests against a surface on its bottom edge that prevents it from moving vertically. A force is applied at the top right corner of the plate. We will use the finite element method we just created to compute the displacements and stresses in the plate.

The plate is made of steel with a thickness of 0.5 inches.

$E = 30 \times 10^6 \; psi$

$\nu = 0.25$




There are two elements and the stiffness matrices for each element can be computed with the equation:

$$k_e = t_e A_e B^T D B \tag{4.66}$$

Where

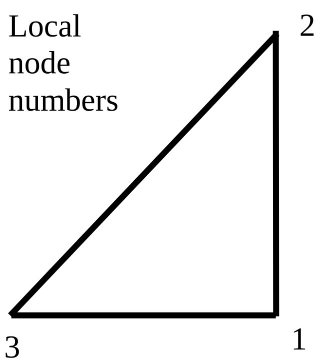
$$D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = 10^7 \begin{bmatrix} 3.2 & .8 & 0 \\ .8 & 3.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix} \tag{4.14}$$

for both elements and

$$B = \frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \tag{4.61}$$

can be applied to each of the elements.

**<u>For Element 1</u>**




$$\det J = x_{13}y_{23} - x_{23}y_{13}$$
$$\det J = 3\times 2 - 3\times 0 = 6$$

$$B_1 = \frac{1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

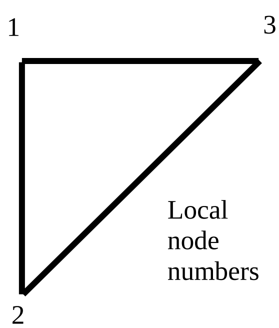
$$k_1 = t_1 A_1 B^T DB = (0.5)(3)B^T DB$$

We use local node numbers to define B. Later on, when we create the global stiffness matrix we will translate these local node numbers to global node numbers.

$$k_1 = 10^7 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 7 & 8 \end{array} \xleftarrow{} \text{DOF} \\ \begin{bmatrix} .983 & -.5 & -.45 & .2 & -.533 & .3 \\ -.5 & 1.4 & .3 & -1.2 & .2 & -.2 \\ -.45 & .3 & .45 & 0 & 0 & -.3 \\ .2 & -1.2 & 0 & 1.2 & -.2 & 0 \\ -.533 & .2 & 0 & -.2 & .533 & 0 \\ .3 & -.2 & -.3 & 0 & 0 & .2 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 7 \\ 8 \end{array}$$

Notice that the stiffness matrix is symmetric.

**<u>For Element 2</u>**




$$\det J = x_{13}y_{23} - x_{23}y_{13} = (-3)(-2) - (-3)(0) = 6$$

$$B_2 = \frac{1}{6}\begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix}$$

DOF

$$k_2 = 10^7 \begin{array}{c} \begin{array}{cccccc} 5 & 6 & 7 & 8 & 3 & 4 \end{array} \\ \begin{bmatrix} .983 & -.5 & -.45 & .2 & -.533 & .3 \\ -.5 & 1.4 & .3 & -1.2 & .2 & -.2 \\ -.45 & .3 & .45 & 0 & 0 & -.3 \\ .2 & -1.2 & 0 & 1.2 & -.2 & 0 \\ -.533 & .2 & 0 & -.2 & .533 & 0 \\ .3 & -.2 & -.3 & 0 & 0 & .2 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \\ 3 \\ 4 \end{array}$$

We combine these two elemental stiffness matrices into a global stiffness matrix by expanding them to 8x8 matrices and adding them. This is the same procedure we used in the finite element analysis of a truss.

DOF

$$K = 10^7 \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} .983 & -.5 & -.45 & .2 & 0 & 0 & -.533 & .3 \\ -.5 & 1.4 & .3 & -1.2 & 0 & 0 & .2 & -.2 \\ -.45 & .3 & .983 & 0 & -.533 & .2 & 0 & -.5 \\ .2 & -1.2 & 0 & 1.4 & .3 & -.2 & -.5 & 0 \\ 0 & 0 & -.533 & .3 & .983 & -.5 & -.45 & .2 \\ 0 & 0 & .2 & -.2 & -.5 & 1.4 & .3 & -1.2 \\ -.533 & .2 & 0 & -.5 & -.45 & .3 & .983 & 0 \\ .3 & 0.2 & -.5 & 0 & .2 & -1.2 & 0 & 1.4 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

We eliminate the DOFs 2, 5, 6, 7, and 8 because the nodes are constrained and there are no displacements associated with these DOFs.

$$
K = 10^7 \begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{bmatrix}
.983 & -.5 & -.45 & .2 & 0 & 0 & -.533 & .3 \\
-.5 & 1.4 & .3 & -1.2 & 0 & 0 & .2 & -.2 \\
-.45 & .3 & .983 & 0 & -.533 & .2 & 0 & -.5 \\
.2 & -1.2 & 0 & 1.4 & .3 & -.2 & -.5 & 0 \\
0 & 0 & -.533 & .3 & .983 & -.5 & -.45 & .2 \\
0 & 0 & .2 & -.2 & -.5 & 1.4 & .3 & -1.2 \\
-.533 & .2 & 0 & -.5 & -.45 & .3 & .983 & 0 \\
.3 & 0.2 & -.5 & 0 & .2 & -1.2 & 0 & 1.4
\end{bmatrix}
\end{array}
\begin{array}{c} \text{DOF} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
$$

Our reduced stiffness matrix becomes

$$
K = 10^7 \begin{bmatrix} .983 & -.45 & .2 \\ -.45 & .983 & 0 \\ .2 & 0 & 1.4 \end{bmatrix}
$$

We are solving the equation

$$
KQ = F
$$

so our problem becomes.

$$
10^7 \begin{bmatrix} .983 & -.45 & .2 \\ -.45 & .983 & 0 \\ .2 & 0 & 1.4 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -1000 \end{Bmatrix}
$$

Solving we get

$$
\begin{aligned}
q_1 &= 1.908 \times 10^{-5} \\
q_3 &= 8.73 \times 10^{-6} \\
q_4 &= -7.415 \times 10^{-5}
\end{aligned}
$$

We can now compute the stress at each node with:

$$
\sigma = DBq
$$

so

Global DOF

$$\sigma_1 = 10^7 \begin{bmatrix} 1.07 & -.4 & 0 & .4 & -1.7 & 0 \\ .27 & -1.6 & 0 & 1.6 & -.27 & 0 \\ -.6 & .4 & .6 & 0 & 0 & -.4 \end{bmatrix} \begin{Bmatrix} 1.9\times10^{-5} \\ 0 \\ 8.37\times10^{-6} \\ -7.415\times10^{-5} \\ 0 \\ 0 \end{Bmatrix} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 7 \\ 8 \end{array}$$

$$\sigma_1 = \{-93 \quad -1136 \quad -62\} \qquad \text{psi}$$

For element 2 the stress is

Global DOF

$$\sigma_2 = 10^7 \begin{bmatrix} -1.07 & .4 & 0 & -.4 & 1.7 & 0 \\ -.27 & 1.6 & 0 & -1.6 & .27 & 0 \\ .6 & -.4 & -.6 & 0 & 0 & 4.4 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 8.73\times10^{-6} \\ -7.415\times10^{-5} \end{Bmatrix} \begin{array}{l} 5 \\ 6 \\ 7 \\ 8 \\ 3 \\ 4 \end{array}$$

$$\sigma_2 = \{93 \quad 23 \quad -297\} \qquad \text{psi}$$