Introduction to Finite Elements

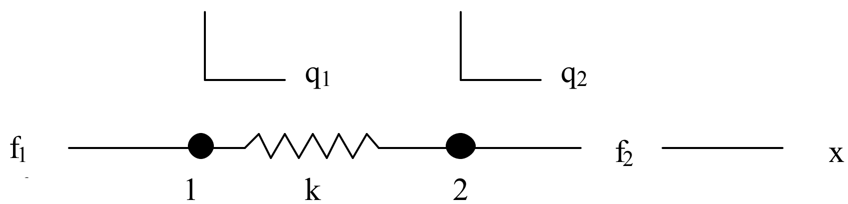
5.0 Introduction

In a previous lecture we looked at computing the forces on elements of a truss. This was relatively easy to do because it was a statically determinate structure. In computing the forces, we did not have to consider deformation of the elements. The truss was a statically rigid structure.

We looked at adding another element to the structure and quickly found it would be statically indeterminate. To solve problems of this nature, we need to look at the deformation of the structure. We will start the development in this lecture.

5.1 Linear Springs

Consider a spring as shown in the following diagram. The variables u_1 and u_2 are the displacements of nodes 1 and 2 respectively.



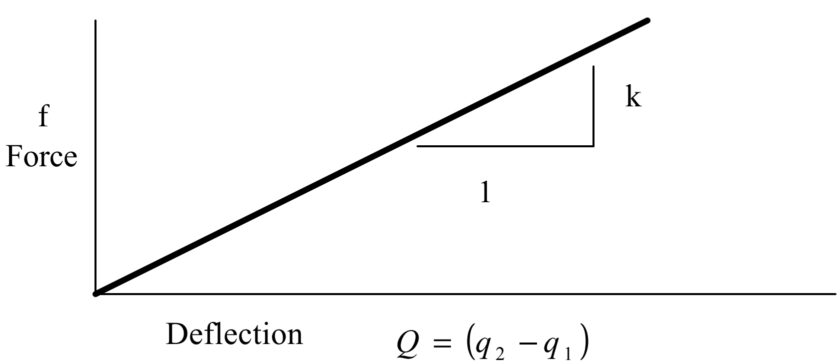
Let

$$Q = q_2 - q_1 \tag{5.2}$$

The force in the spring becomes

$$f = kQ = k(q_2 - q_1) \tag{5.2}$$

This is shown in the graph below.
For equilibrium we know that



$$f_1 = -f_2 \tag{5.3}$$

We can rewrite 5.2 as:

$$\begin{aligned} f_1 &= -k(q_2 - q_1) \\ f_2 &= k(q_2 - q_1) \end{aligned} \tag{5.4}$$

From our previous lectures we know that these can be written in matrix form as:

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{5.5}$$

or in general as:

$$[K]\{Q\} = \{F\} \tag{5.6}$$

Where

$$[K] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \tag{5.7}$$

We call this the element stiffness matrix. Note that it is a symmetric matrix since the upper right off diagonal term(s) is the same as the lower left off diagonal term(s). This indicates the structure is linearly elastic and there is symmetry in the movement of the nodes. A small movement in node 1 produces the same force in the spring as the same small movement in node 2 in the opposite direction.

We could stop here and try to solve equation (5.5) for the forces on each end of the spring but we will run into problems. The stiffness matrix is singular. We know this because casual inspection show us that

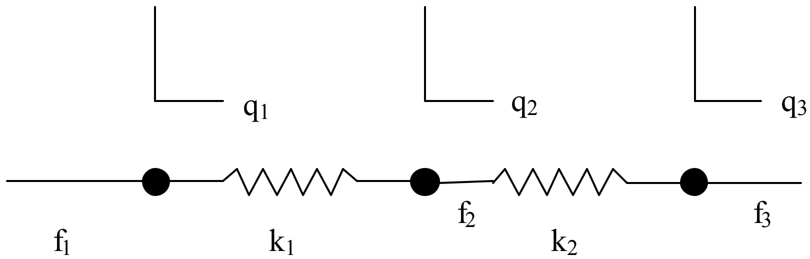
$$\det|K| = 0 \tag{5.8}$$

The matrix is singular because there is nothing anchoring the spring. We can tie the structure down so that node 1 will not move and this will make the system solvable.

5.2 Multiple Springs

The previous example was very simple and we did not need the complexity of the matrix solution. As the problems become more complex, we can use the matrix (finite element) approach to our benefit.

Consider



We can use the technique developed previously to write the matrix equations for the two springs. For the first spring we have equations for node 1 and 2 we have the equation. Note that the force subscript is the force acting on the node and the force superscript is the spring on which the force is acting.

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 \end{Bmatrix} \tag{5.9}$$

For the second spring we have equations for node 2 and 3

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} f_2^2 \\ f_3^2 \end{Bmatrix} \tag{5.10}$$

We can expand both sets of equations to include all three nodes by adding a row and column of zeros in the appropriate position. Equation (5.9) becomes

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 \\ 0 \end{Bmatrix} \tag{5.11}$$

and equation (5.10) becomes

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_2^2 \\ f_3^2 \end{Bmatrix} \tag{5.12}$$

The row and column of zeros has not effect on the equations. You can quickly determine this by multiplying through to create the system of equations. In both (5.11) and (5.12) you will get a 0 = 0 equation which we know to be correct.

We can combine the two sets of equations by adding the terms cell as shown below.

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 + f_2^2 \\ f_3^2 \end{Bmatrix} \tag{5.13}$$

Making the substitutions $F_1 = f_1^1$, $F_2 = f_2^1 + f_2^2$, and $F_3 = f_3^2$ we have

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \tag{5.14}$$

which is in the basic form of

$$[K]\{Q\}=\{F\} \tag{5.15}$$

Note that [K] is the stiffness matrix and that it is singular since on restraints have been applied.

5.3 Example

Given a two spring system as that shown above, let k_1 = 50 lb/in, k_2 = 75 lb/in, $F_2 = F_3$ = 100 lbs. We will attach node 1 to a stationary frame so that its displacement, q_1 will be zero. We need to compute q_2 and q_3.

Substituting into equation 5.14 yields

$$\begin{bmatrix} 50 & -50 & 0 \\ -50 & 125 & -75 \\ 0 & -75 & 75 \end{bmatrix} \begin{Bmatrix} 0 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 100 \\ 100 \end{Bmatrix}$$

Note that u_1 is zero so we replace it with zero in the displacement vector. We do not know the force at node 1 so it becomes a reaction. Multiplying through by the top row of [K] results in:

$$-50q_2 = F_1$$

which we allows us to solve for F_1 when we know u_2.

Knowing that q_1 is zero, allows us to eliminate the first row and column from the matrix and vectors. Our problem becomes:

$$\begin{bmatrix} 125 & -75 \\ -75 & 75 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 100 \end{Bmatrix}$$

We solve this for the displacements at nodes 2 and 3. The solution yields $q_2 = 4.0$ and $q_3 = 5.3333$.