An Agent-Based Adaptation of Friendship Games: Observations on Network Topologies

David S. Dixon

University of New Mexico, Albuquerque NM 87131, USA

Abstract. A friendship game in game theory is a network game in which a player's immediate neighbors on the network are considered friends. Two friendship-based game models are examined: strategic complements and strategic substitutes. Strategic complements represent decisions for which it is preferable to do what one's friends are doing, such as adopting a common software product. Strategic substitutes represent decisions for which it is preferable to let one friend act alone, such as the private provision of a public good. The game theory models predict the rate of convergence to specific equilibrium outcomes for each model. This paper employs an agent-based model (ABM) implementation of friendship games to examine the sensitivity of equilibrium states to network topology. In future work, the ABM model can provide a means to examine the motivations for behaviors of specific individuals in these models beyond closed-form payoff functions.

1 Introduction

Lamberson [7] presents a network-game model of the influence that friends - defined as immediate neighbors on a network - have on individual preferences and the effect this has on long-run equilibrium. Friendship games are applicable to problems for which peer choice is important. Examples include the adoption of standards or common tools, such as word processing software. Additionally, friendship games find use in problems of free-riding, such as the private provision of a public good like a street light or a web server. Lamberson [7] shows that each of these reach one or two distinct equilibrium states. This paper examines how network topology affects those equilibrium states using an agent-based model (ABM) developed in NetLogo [9] for this purpose.[3][1]

[1] The NetLogo model is available for download from the NetLogo User Community Models library http://ccl.northwestern.edu/netlogo/models/community/ and from OpenABM http://www.openabm.org/model/2661.

2 Network Games

Galeotti et al. [5] present the theoretical basis for, and some examples of, network games. In these games, the players are distributed on a random network and the payoffs are functions of the expressed preferences of the immediate neighbors on the network. For a model of strategic substitutes, the payoff is such that, if the neighbors are paying the cost, the player has no incentive to also pay it. This is a free-rider model, similar to the private provision of a public good. For a model of strategic complements, the payoff is highest for the choice that is supported by multiple neighbors. This is similar to a network externality, where adopting the most common word processing software, for example, maximizes the ability to share documents with neighbors. Lamberson [7] adopts the term *friend* for these network neighbors, reflecting the fact that adjacent nodes in a social network can be quite distant geographically. The notation in this section is taken from [7].

2.1 The Strategic Complements Model

Suppose there are two strategies, x and y. If an agent has k friends, then, at any given instance, there are k_x of them playing strategy x, and k_y of them playing strategy y. For the strategic substitutes models, the payoff for playing strategy x is

$$\pi_x\left(k_x\right) = f\left(k_x\right) - c_x \tag{1}$$

and the payoff for playing strategy y is

$$\pi_y\left(k_x\right) = f\left(k - k_x\right) - c_y \tag{2}$$

where f is a non-decreasing function and c_x and c_y are the costs of playing x and y, respectively.

The adoption of a standard is a strategic complement: an agent chooses what most of its friends choose. A decision to adopt a strategy has a positive affect in that friends tend to take the same choice as the agent.

2.2 The Strategic Substitutes Model

For the strategic substitutes models, the payoff for playing strategy x is

$$\pi_x (k_x) = 1 - c_x \tag{3}$$

where $0 < c_x < 1$ and the payoff for playing strategy y is

$$\pi_y (k_x) = \begin{cases} 1 & k_x \geq 1 \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

The provision of a public good is a strategic substitute: an agent needn't provide it unless none of its friends do. The decision to adopt a strategy has a negative effect in that friends tend to take the opposite choice of the agent.

3 Approximating a Random Network

The models in [7] feature 1000 players on a random network. In some cases it is a regular random network, in others it is a Bernoulli random network with an edge probability of 0.01. That is, for a network potentially connecting all players to all players, there is a probability of one in one hundred that a given connection will actually be there. The number of other players to which a player is connected is that player's *degree*. The average degree for this random network is approximately 10. That is, players have, on average, ten friends.

In the ABM developed for this paper, there are fours ways in which a random network can be generated. These are referred to as the *regular*, *Erdős-Rényi*, *Gilbert,* and *preferential attachment* network models. The following are descriptions of these network models.

3.1 The Regular Random Network Model

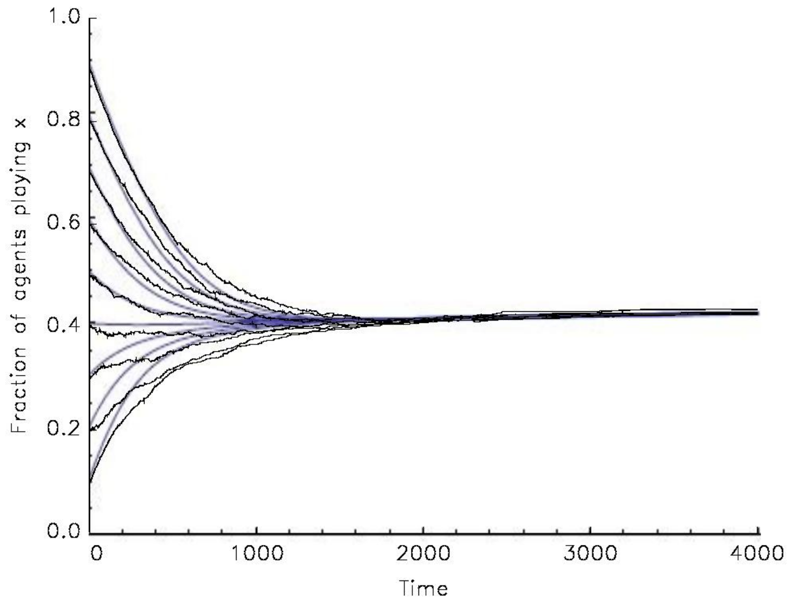
One way to form a random network is for each agent to make two friends, but only with other agents that don't already have two friends. This results in a regular random network where the nodes have a uniform degree of two, ensuring that the average degree is two. It is a high connectivity network: no agents will end up completely disconnected from the network.

For example, a simple form of the strategic substitutes model in Sect. 2.2 is implemented in NetLogo and simulated as outlined in Sect. 3.5. Figure 1a shows the results for a degree 2 regular network of 1000 nodes. This plot overlays the ABM results (black) on an image of the corresponding numerical results in [7] (blue). As with the numerical result, about 40 percent of the nodes in the ABM are playing strategy x at equilibrium, and the ABM takes slightly longer to reach equilibrium than the numerical model.

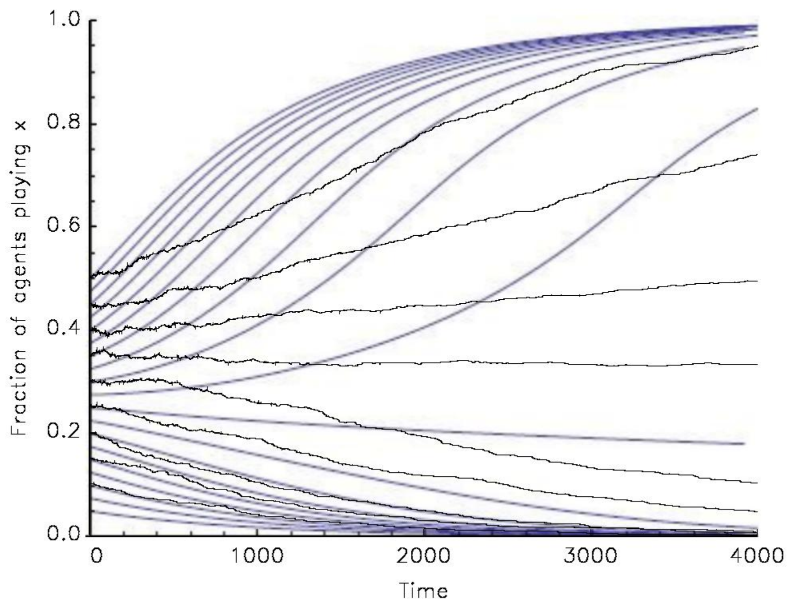
Similarly, a simple form of the strategic complements model in Sect. 2.1 is implemented in NetLogo and simulated as outlined in Sect. 3.5. Figure 1b shows the results for a degree 2 regular network of 1000 nodes. This plot overlays the ABM results (black) on an image of the corresponding numerical results found by [7] (blue). In the numerical model, an initial distribution of 22.5 percent or fewer playing x move to an *all-out* (no players playing x) equilibrium. Similarly, initial distributions with 30 percent or more playing x move to an *all-in* (all players playing x) equilibrium. In the ABM results, an initial distribution of 40 percent playing x appears to be moving to the all-in equilibrium, an initial distribution of 30 percent playing x go to the all-out equilibrium, and an initial distribution of 35 percent playing x is decreasing slowly but monotonically. Here the ABM results differ considerably from the numerical results. The split between the all-in and all-out equilibria is evident but at a higher initial distribution in the ABM, and the time for the ABM to reach equilibrium is much greater. The differences are not yet understood.

3.2 The Erdős-Rényi Random Network Model

A Bernoulli random network with n nodes and probability p that an edge will exist results in nodes with degrees that are binomially distributed about the mean np.[4] There are two common models for this type of network. One is the the *G(n, M)* model, which is called the *Erdős-Rényi random network model* in this paper, and the other



(a) Strategic substitutes, cost $c_x = 0.5$.



(b) Strategic complements, cost ratio $c_x : c_y = 50 : 50$.

Fig. 1: Degree two regular network overlaid on [7] Figs. 2 and 1, respectively.

is the *G(n, p)* model, which is called the *Gilbert random network model* in this paper (see Sect. 3.3).

In the *G(n, M)* model, a network is chosen at random, with uniform distribution, from the collection of all possible networks with n nodes and M edges. For the models in this paper, M is not known a priori, so a *G(n, M)* model is approximated by adding edges between randomly chosen pairs of nodes until the mean degree reaches the desired value.

3.3 The Gilbert Random Network Model

If all friendship pairs are equally probable with probability p then a Gilbert random network, *G(n , p)* is formed.[6] The mean degree is np, where n is the number of nodes. For the models in this paper, this is created with nested loops: an outer loop over a randomized list of all nodes, and an inner loop over a randomized list of all nodes that haven't already come up in the outer loop. In the inner loop, a connection is formed if a uniform random draw is less than or equal to p.

3.4 The Preferential Attachment Network Model

The preferential attachment model [10] is included in the NetLogo [9] demo library and is based on an approach by Barabási and Albert [2]. This is an approximation of a *scale-free network,* a network with a power-law distribution of node degree. Also called a Pareto distribution, it results in a few nodes having a very large number of connections and many nodes with only one connection. Scale-free networks are seen in academic citations [8] and in a variety of Internet linkages.[1] Albert and Barabási [1] found that the probability of a link for a node with degree k is $p(k) = \alpha k^{-\gamma}$where γ is between 2 and 3. The NetLogo preferential attachment algorithm yields a γ of approximately 2, and a mean degree of approximately 2.

3.5 The ABM Simulation

In order to make a direct comparison with the numerical models in [7], the NetLogo models update a single, randomly selected agent at each time step. This random sampling means that, for a network with 1000 nodes, in the first 1000 time steps, some agents may not be updated at all, and others may be updated more than once.

These are the steps in a simulation:

1. Randomly assign agents an initial strategy. Each run involves nine simulations, each with a different initial strategy distribution:
   (a) For strategic substitutes, starting initial distributions of the fraction of agents playing strategy x are 10 percent through 90 percent in steps of 10 percent.
   (b) For strategic complements, starting initial distributions of the fraction of agents playing strategy x are 10 percent through 50 percent in steps of 5 percent.
2. Each time step, a node is selected at random and that node selects a strategy based on the payoffs. This may be the same as the strategy already being played.
3. Each simulation proceeds for 4000 time steps, except as noted.

3.6 The Random Network Models

The following models use random networks with 1000 nodes and the following payoff functions, suggested in [7]. For a strategic substitute, the payoff for playing x is positive if four or fewer neighbors are playing x, and the payoff is zero otherwise:

$$\pi_x^{substitute} = \begin{cases} 1 & k_x \leq 4 \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

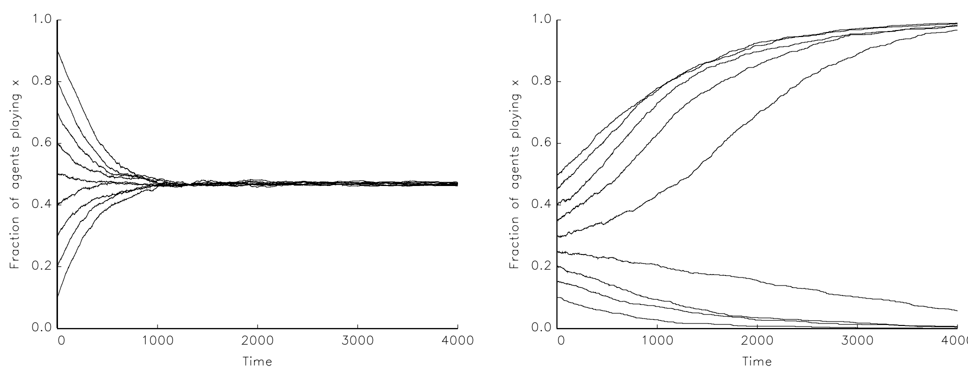
For strategic complements, the payoff for playing x is positive if four or more neighbors are playing x, and the payoff is zero otherwise:

$$\pi_x^{complement} = \begin{cases} 1 & k_x \geq 4 \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

In either case, the payoff for playing y is one less the payoff of playing x:

$$\pi_y = 1 - \pi_x.$$

Degree 10 Regular Random Network. Plots of the ABM results with a degree 10 regular random network are shown in Fig. 2. The equilibrium for strategic substitutes is between 46.6 percent and 47.9 percent playing x. For strategic complements, the all-in or all-out division is between 25 percent and 30 percent playing x.



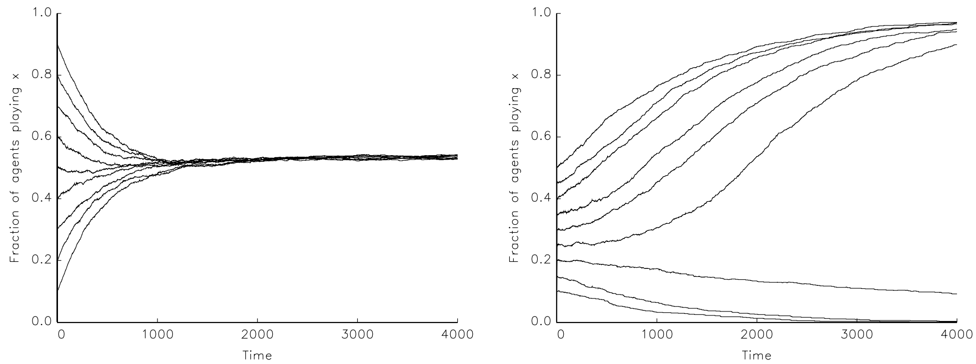
(a) Strategic substitutes.

(b) Strategic complements.

Fig. 2: Degree 10 regular random network.

Degree 10 Bernoulli Random Network. Plots of the ABM results with a degree 10 Bernoulli random network are shown in Fig. 3. This is an Erdős-Rényi random network, but the results for a Gilbert random network are effectively identical. The equilibrium for strategic substitutes is between 53.0 percent and 54.3 percent playing x. For strategic complements, the all-in or all-out division is between 20 percent and 25 percent playing x.

Degree 2 Preferential Attachment Network. Plots of the ABM results with a degree 2 preferential attachment random network are shown in Fig. 4. Note that the horizontal axis goes to 10000 time steps for the strategic complements plot only. The preferential attachment network can only yield a mean degree of 2, so a degree 10

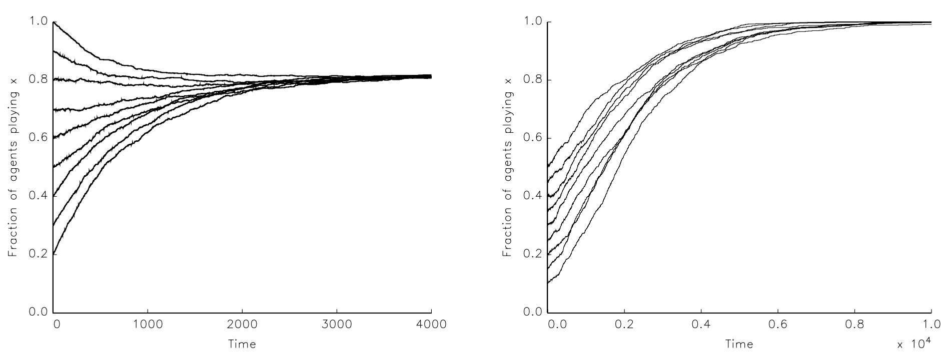


(a) Strategic substitutes.

(b) Strategic complements.

Fig. 3: Degree 10 Bernoulli random network.

network is not attainable with this network model. The strategic substitutes curves converge to an equilibrium between 80.7 percent and 81.8 percent playing x at about 3100 time steps. For strategic complements, equilibrium isn't reached until about 8000 time steps. For this range of starting values, only the all-in equilibrium is reached. Further investigation implies that all-in is the only equilibrium for starting values as low as 1 percent playing x.



(a) Strategic substitutes.

(b) Strategic complements.

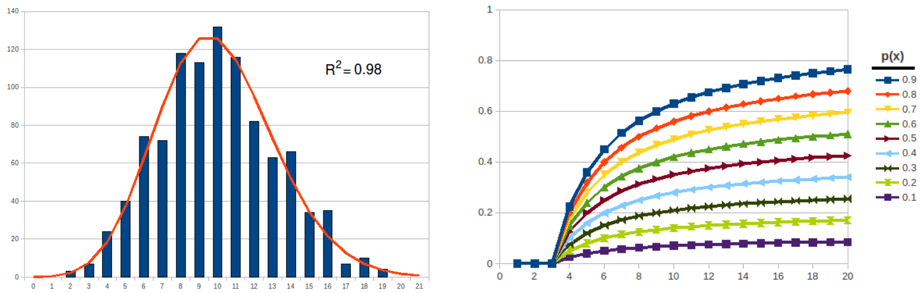
Fig. 4: Degree 2 preferential attachment ABM with $n = 1000$

4 Discussion

The intriguing dynamics of stable equilibria in friendship games is established numerically in [7]. The correspondence of the ABM to the numerical model is demonstrated in [3]. What is shown in this paper is that the equilibrium values are affected by network topology. For the same strategic substitute payoff, the fraction playing x at equilibrium is higher for a degree 10 regular network than for a degree 2 regular network. Furthermore, the fraction playing x at equilibrium is higher for a Bernoulli network than for a regular network, and higher still for a power-law network. For the same strategic complement payoff, the division between the all-in and all-out equilibria is lower for a degree 10 regular network than for a degree 2 regular network. Additionally, the division between the all-in and all-out equilibria is lower for a Bernoulli network than for a regular network, and appears to go to zero for a power-law network.

The trend in the equilibrium for strategic substitutes is a result of introducing increasing numbers of agents with only a few friends. For example, an equilibrium of 40 percent playing x means that 60 percent of the agents are free-riding. The distribution of degree in a sample degree 10 Bernoulli random network is shown in Figure 5a. The probability that an agent is free-riding as a function of degree and $p(x)$, the current probability of playing strategy x, is shown in Figure 5b. Note the asymmetry in the probability of free-riding: below degree 4, the zero probability of free-riding means that all agents are playing strategy x, while the agents in the upper tail of the distribution have a decreasing probability of playing strategy x. The peak in the power-law distribution is at degree 1, causing this effect to be especially pronounced.

The trend in the equilibria for strategic complements is less obvious. The probability of free-riding shown in Figure 5b is the probability that an agent is not playing strategy x and at least four friends are. This is also the condition for an agent to switch to playing strategy x as a strategic complement. For a Bernoulli random network, the probability of higher-degree agents switching to strategy x is greater than the mean, so that lower values of $p(x)$ lead to the all-in equilibrium as compared with a regular network. The apparent single all-in equilibrium for a power-law network is not yet under-



(a) Degree distribution. (b) Probability of free-riding.

Fig. 5: Degree 10 Bernoulli random network.

stood, but may be an artifact of the mean degree of 2 inherent in the preferential attachment network.

5 Future Work

The payoffs in a friendship game ABM are not constrained to closed-form mathematical functions, and can incorporate adaptive behaviors such as learning and heuristics. These could enable the construction of models of voters, economic agents, or decision-makers in which the payoff (or utility or fitness) depends on the preferences of multiple groups of friends over multiple conflicting issues. For example, voters may be influenced by a workplace network on issues relating to their livelihood, and by a very different social network on other topics. Another example is a producer which shares one kind of network with consumers, and another kind of network with competitors. Some issues or events may result in evolving network topology, such as emerging social movements, natural disasters, and economic upheaval. Another area of interest is intentional network disruption.

6 Conclusion

Network topology has a significant effect on the equilibrium outcome in a network game. Although the equilibrium modes differ based on whether the payoff represents a strategic complement or a strategic substitute, the equilibrium values are influenced by the degree and topology of the network. The implication is that the results from network game models may only be valide in the context of the network topology under which they are developed. Efforts to apply network game models to social and other networks must begin with an accurate assessment of the topology of the target network, and a thorough understanding of how the results of the model are affected by network topology.

References

1. Albert, R., Barabási, A.: Statistical mechanics of complex networks. Reviews of modern physics 74(1), 47 (2002)
2. Barabási, A., Albert, R.: Emergence of scaling in random networks. Science 286(5439), 509 (1999)
3. Dixon, D.S.: Preliminary results from an agent-based adaptation of friendship games. 86th Annual Conference of the Western Economics Association International (2011)
4. Erdős, P., Rényi, A.: On random graphs, I. Publicationes Mathematicae (Debrecen) 6, 290–297 (1959)
5. Galeotti, A., Goyal, S., Jackson, M., Vega-Redondo, F., Yariv, L.: Network games. Review of Economic Studies 77(1), 218–244 (2010)
6. Gilbert, E.: Random graphs. The Annals of Mathematical Statistics pp. 1141–1144 (1959)
7. Lamberson, P.: Friendship-based Games. 37th Annual Conference of the Eastern Economics Association (2011)
8. de Solla Price, D.: Networks of scientific papers. Science 149(3683), 510 (1965)
9. Wilensky, U.: Netlogo (1999), http://ccl.northwestern.edu/netlogo/, accessed 31 August 2011
10. Wilensky, U.: Netlogo preferential attachment model (2005), http://ccl.northwestern.edu/netlogo/models/PreferentialAttachment, accessed 31 August 2011