Chapter 5. Mechanics

5.1. Introduction

Dogmatic presentations sometimes generate the perception that science is completely logical and understood. This short-sighted view ignores the progress that continues to refine and revolutionize our conceptualization of the natural world. We will trace some of that progress here.

We cannot hope to give more than a superficial treatment of the deep and broad subject of physics. Our goal is to present those aspects of physics that will benefit an understanding of introductory chemistry. In the interest of space we will emphasize only the most significant principles and insights of individuals while unjustly minimizing the pathfinding contributions made by many others.

Debates on the nature of existence that the earliest thinkers could not resolve will continue into the future. Is matter continuous or discrete? Is the continuity of numbers real or merely imagined? Is the universe holistic or reducible? Is the whole greater than the sum of its parts? A sample of matter such as a cloud appears continuous on a macroscopic scale, but passage through a fog suggests that the matter is divisible. Does the division ever end? What are the consequences if it does and what if it doesn't? The searches for more ‘‘fundamental particles’’ and for unifying principles represent the modern attempt to explore the nature of matter. As we shall see, the fundamental opposition between the notions of continuous and discrete existence is a recurring theme in science.

5.2. Particle Mechanics

The studies of the structures and motions of material systems are called *statics* and *dynamics*, respectively, and together form the subject of **mechanics**. The basic laws of the mechanics of macroscopic objects, called *classical mechanics*, were first clearly stated by Isaac Newton[1] in *Principia Mathematica*, published in 1787, and form the cornerstone of physics.

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[1] Sir Isaac Newton (British, 1642-1727), who weighed less than three pounds at birth on Christmas day, returned to his birthplace at his mother's farm to escape the plague of 1666, where he conceived of the laws of gravitation as applying universally to all objects. His law explained the astronomical laws of planetary motion derived by Keppler from the observations of Copernicus, and his friend Edmund Halley used the law to predict

the return of a comet. Paradoxically, Newton, whose object was to prove the existence of God (his religious writings exceed those of his scientific writings), laid the foundations of a mechanistic, atheistic explanation of nature.

Newton's occult hermetic and alchemical pursuits of element transmutation led to the explosive destruction of his laboratory at Trinity College in 1694. An atomist (believer in the discrete nature of matter), he elucidated the nature of light using both corpuscular (particle) and wave models. This, together with his vacillation throughout his career between action at a distance versus action through an intervening corpuscular aether presaged the Twentieth Century notion of duality of existence.

Classical mechanics is founded on a model of finite particles moving in an absolute and independent framework of space and time.[2] The structure and motion (i.e. the "state") of particles of matter are described in terms of *coordinates*, or positions of the component parts of position $\mathbf{r}(t)$ relative to a point in space called the *origin* and distances and directions from the origin. The bold notation serves to denote an array of component parts, $\mathbf{r} = [r_i]$, or in vector notation $\vec{r} = \sum r_i \vec{e}_i$, where $\vec{e}_i$ is the unit length along an axis. A *Cartesian coordinate system* uses orthogonal (perpendicular) lines (*axes*) to measure the distances and directions of a point relative to the origin (cf. Section 3.3). Directions may also be measured in terms of a distance from the origin (*radius*) and angles of rotation about axes in *polar coordinate systems* (Fig 3.1). In three dimensions $\mathbf{r} = [x, y, z] = [r, \theta, \phi]$, or $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = r\vec{e}_r + \theta\vec{e}_\theta + \phi\vec{e}_\phi$. Polar coordinate systems are useful for describing the dynamics of particles which interact along lines connecting them together.

Motion adds the dimension of time to space and is measured in terms of changes, or differences (cf. Section 3.11). The rate of change of (linear or angular) distance with time is a measure of the *velocity* of an object, specifically

$$\boxed{\mathbf{v} = \dot{\mathbf{r}} \equiv \frac{d\mathbf{r}}{dt} \qquad (5.1)}$$

Note that instantaneous derivatives are related to tangent slopes and finite differences refer to average values, as discussed in Section 3.3. Newton and Leibnitz invented the calculus to describe the motion of systems. The dot notation represent derivatives is Newton's while the d notation is Liebnitz'. Speed is the magnitude of velocity which we denote by symbol v to avoid ambiguity with the SI symbol for seconds, s. The components of velocity in Cartesian coordinates are $[\dot{x}, \dot{y}, \dot{z}]$ whereas in polar coordinates they are $[\dot{r}, \dot{\theta}, \dot{\phi}]$.

Acceleration is defined as the change in velocity during a time interval,

[2] "Give me matter and motion, and I will construct the universe," said Descartes. Modern physics has modified all these notions, as discussed below.

$$\boxed{\mathbf{a} = \dot{\mathbf{v}} \qquad (5.2)}$$

Note that acceleration can be due to a change in velocity direction as well as change in velocity magnitude. Thus a particle moving at constant speed along a curve, such as a swinging rock tied to one end of a string, experiences an acceleration due simply to the change in *direction* of the velocity. If r is the radius of curvature of the curve (e.g. radius of a circle), it can be shown that the magnitude of the acceleration due to change in direction of velocity is $a = \frac{v^2}{r}$. Any change in magnitude of angular velocity would add to this term.

Example 5.1 In 1904 Henry Ford drove an automobile one mile in 39.40 seconds. What was his average speed in mi/hr?

Speed is velocity without regard to direction. His speed probably wasn't constant, but the definition of velocity can be used to calculate the average speed of his run. Converting the units gives

$$\text{average speed} = \frac{\Delta d}{\Delta t} = \left(\frac{1\ \text{mi}}{39.40\ \text{s}}\right) \times \left(\frac{60\ \text{s}}{1\ \text{min}}\right) \times \left(\frac{60\ \text{min}}{1\ \text{hr}}\right) = 91.37\ \frac{\text{mi}}{\text{hr}}$$

Not bad for 1904!

Example 5.2 Compare the average acceleration of an automobile which accelerates uniformly from 0 to 60 mi/hr in 6.3 s to the acceleration of gravity of a freely falling object, 32 ft/s^2.

From the definition:

$$\text{average acceleration} = \frac{\Delta v}{\Delta t} = \left(\frac{60\ \text{mi/hr} - 0\ \text{mi/hr}}{6.3\ \text{s}}\right) \times \left(\frac{5280\ \text{ft}}{1\ \text{mi}}\right) \times \left(\frac{1\ \text{hr}}{3600\ \text{sec}}\right) = 14.0\ \frac{\text{ft}}{\text{s}^2}$$

or about 0.44 "g".

Mechanics formulates the laws of motion in terms of resistance to change in amount or direction of motion, called *inertia*. The amount of inertia of a body is measured by a quantity called *mass* (m) for linear motion, and *moment of inertia* (I) equal to mass times distance to the origin squared for angular motion. The central quantity of Newtonian motion is *momentum*, p, defined for both linear and angular motion as the product of inertia and velocity. Common symbols are p (somewhat ambiguous) and l for linear and angular momentum, respectively. Thus

$$\boxed{\begin{aligned} \mathbf{p}_{\text{linear}} &\equiv m\mathbf{v}_{\text{linear}} \equiv \mathbf{p} && (5.3a) \\ \mathbf{p}_{\text{angular}} &\equiv \mathbf{I}\mathbf{v}_{\text{angular}} \equiv \mathbf{l} && (5.3b) \end{aligned}}$$

The fundamental law of a body which resists acceleration due to inertial mass is Newton's "Second Law of Motion", or definition of force as the rate of change of momentum (angular force, called *torque* may be denoted by n)

$$\boxed{\mathbf{f} = \dot{\mathbf{p}} \qquad (5.4)}$$

The Newtonian mechanics program is: identify the functional forms for the forces acting on a body and solve the (differential) equations of motion to determine positions as a function of time (*trajectories*). For bodies whose mass remains constant, Newton's Law of linear motion takes the famous form f = ma.[3] When no forces act on a body, momentum remains constant (according to the rule that the derivative of a constant is zero), and a constant-mass system continues its course with constant velocity. This special case of Newton's Second Law is called Newton's First Law, but was inherited by Newton from his predecessor, Galileo.[4]

Systems are classified according to the *forms* of the forces they experience. Simplifying assumptions in particle mechanics are that interactions somehow act over space ("action at a distance"), and that the interactions of complex systems can be broken down into interactions

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[3] Since $\mathbf{f} = \frac{d\mathbf{p}}{dt} = \frac{dm\mathbf{v}}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}$ according to the rules for the derivative of a product and the derivative of a constant as discussed in Section 3.11.

[4] Galileo Galilei (Italian, 1564-1642) deduced that freely falling objects fall with constant acceleration independently of mass, that swinging pendulums could measure constant time intervals independent of amplitude, and that the Milky Way consists of a galaxy of stars independent of our solar system. His invention called the telescope permitted him to remark "I now have visual proof of what I already knew through my intellect," but threatened the clergy who refused to view a potentially imperfect heaven, which led to a crisis between the Church and Galileo. The Church inquisitioned, Galileo recanted, and lived out his last years under house arrest. Newton paid tribute to Galileo (and Descartes and Keppler) in his laconic comment to his rival Robert Hooke, "If I have seen further than you and Descartes it is by standing upon ye sholders of Giants."

between two particles at a time.[5] Three examples of central (no angular dependence) forces having simple mathematical forms (variable and parameter dependencies) deduced from observations are Isaac Newton's 1666 *Law of Gravitational Attraction* of inertial bodies, Robert Hooke's 1678 *Law of Elastic Force* of springs and Charles Coulomb's 1784 *Law of Electrostatic Force* of charged particles.

Newton's Law of Gravitation has the mathematical form that the force is proportional to product of the masses (m) of two bodies (the parameters), and falls off proportional to the square of the separation (r) between them (the variable).

$$\boxed{f_{gravitational} = -G\,\frac{m_1 m_2}{r_{12}^2}\,, \qquad (5.5)}$$

where G is an universal constant equal to 6.67259×10^{-11} N $m^2 kg^{-2}$ called the gravitational constant.[6]

Example 5.3 Deduce the mass of the earth from knowledge of the magnitudes of the acceleration of objects at the surface of the earth and the earth's radius.

An object of mass m at the surface of the earth experiences a mean acceleration of $g = 9.80665$ m/s^2. The mean radius of the earth at the equator is $R_{earth} = 6378.140$ km. Of course the earth is not exactly spherical, due in part to expansion of at the equator from rotational centrifugal forces of 21 km (.33%) over that at the poles. The equitorial radius represents that of the majority of the earth, however. In identifying R_{earth} with r_{12} in Eq. 5.5, an assumption has been made that the earth acts as a single central particle of mass equal to that of the entire earth.[7]

Applying Newton's inertial law to gravitation:

$$f = ma = mg = \frac{GM_{earth}m}{R_{earth}^2}$$

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[5] *Holistic* descriptions view all parts of the universe as internetworked and interacting simultaneously.

[6] The experimental value of G may be determined directly by measuring the force of attraction of two objects of known masses.

[7] In fact an object at the surface of the earth experiences gravitational attraction to all the particles distributed throughout the earth. Newton struggled with this assumption for some time, and fi nally justifi ed it with a proof employing his recently invented method of calculus.

we find ($1\ \mathrm{N} = 1\ \mathrm{kg\ m\ s^{-2}}$)

$$M_{earth} = \frac{gR_{earth}^2}{G} = 5.979 \times 10^{24}\ \mathrm{kg}$$

to the accuracy of the equitorial radius approximation.

Coulomb's Law of Electricity shares a common mathematical form with gravitation, with charges (q) replacing masses:

$$\boxed{f_{electrical} = k_e \frac{q_1 q_2}{r_{12}^2}, \qquad (5.6)}$$

where $k_e = \dfrac{1}{4\pi\varepsilon_0}$, and $\varepsilon_0 = 8.85418 \times 10^{-12} \mathrm{C^2/(Nm^2)}$ in SI units.[8] While gravitational force is always attractive (since mass is always positive), electrical force can be attractive (for charges having opposite signs) or repulsive (for charges having the same sign).

Example 5.4 Find the ratio of electrostatic to gravitational interaction between two electrons ($\mathrm{mass}_{electron} = 9.10939 \times 10^{-31}$ kg, $\mathrm{charge}_{electron} = 1.602177 \times 10^{-19}$ C).

From Coulomb's and Newton's Laws:

$$\frac{f_{electrical}}{f_{gravitational}} = \frac{kq_{electron}^2}{Gm_{electron}^2} = 4.1667 \times 10^{42}$$

and we see that gravitational forces are negligible compared to electrostatic forces for the smallest charged particles. Classical mechanics assumes that properties of matter like mass and charge are additive (the total amount is the sum of the part amounts). This means that while atoms and molecules may be held together by net attractive electrical forces of their subatomic particles, macroscopic matter is usually electrically neutral, leaving gravitational forces (which are only attractive and accumulative) as the dominant interaction.[9]

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[8] The unit of charge in SI units is the Coulomb (C), defined as one Ampere times one second (C = As). Other systems of units (Gaussian units) set k to unity.

[9] A counterexample is a toy balloon which sticks to a wall after being rubbed against a piece of clothing. Sufficient electrical charge is transferred between the balloon and clothing to render an imbalance which, in

Hooke's Law of Elastic Force simply states that the restoring force of a spring is proportional to the stretch or *displacement* r from a resting, or *equilibrium* position of the spring:

$$\boxed{f_{spring} = -k_s r, \qquad (5.7)}$$

where k_s is the *restoring force proportionality constant*, a parameter characteristic of the system. An inertial system (mass) with elastic restoring force experiences an harmonic (oscillatory) motion.

Example 5.5 Analyze the motion of a mass attached to a spring which is described by Hooke's Law that the restoring force is proportional to the stretch of the spring.

From Newton's Law:

$$f = \dot{p} = m\dot{v} = m\ddot{r} = -kr$$

where m, the mass of the spring, and k are parameters characteristic to the physical properties of the spring. Dividing the last equality by m on both sides, we are led to seek a function whose curvature (the double dot signifies second derivative with respect to time) is proportional to itself. The trigonometric sin and cosine functions have this property, for, as Table 3.1 shows, the first derivative of one of these is converted into the other (with a possible sign change) and thus the second derivative converts it back again.[10] Sinusoidial motion is *periodic*, in that a given amplitude is repeated periodically after a certain time interval called the period. Such a system is called a *simple harmonic oscillator*, and plays an important role as a model for periodic motion. This model is developed in Section 4.4. The general motion solution to Newton's equation is

$$f(t) = A \sin\left(\sqrt{\frac{k}{m}}\, t\right) + B \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

where A and B are arbitrary constants determined by the initial conditions of amplitude f and velocity $\dot{f}$ of the oscillator at t = 0. For example, a spring stretched out to a value

turn, induces an imbalance in the wall (polarization) with a resulting attraction to the wall sufficient to overcome the gravitational attraction of the balloon to the earth.

[10] The exponential function with imaginary argument also satisfies the equation of motion. The various solutions are related by Euler's identity $\exp ix = \cos(x) + i \sin(x)$.

of B units and released from zero velocity at time zero vibrates with time as[11]

$$f(t) = B \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

Since $\sin(\theta + 2\pi) = \sin(\theta)$ the amplitude returns to its previous value after period of time τ (Greek tau) equal to $2\pi\sqrt{m/k}$. The *linear frequency* of oscillation ν (Greek nu) is the inverse of the period of oscillation. The *angular frequency* of oscillation ω (Greek nu) equals $2\pi\nu$

$$\boxed{\omega = 2\pi\nu = \frac{2\pi}{\tau} = \sqrt{k/m} \qquad (5.8)}$$

5.3. Energy

Invariance or constancy of a quantity is referred to as *conservation* of the quantity. For example, the law of conservation of momentum means that momentum remains constant with time. Conservation reflects an appealing basic symmetry of nature generates sweeping generalizations regarding behavior. Formulating problems to reveal conserved quantities (for example by coordinate transformation) reduces them to simpler problems with straightforward solutions.

There is a price to pay for everything, and in a cause and effect there is a loss of potential in that which causes the effect equivalent to the gain of potential in that which is affected. In mechanics, **energy, E** is the currency of change and measures the ability to induce motion in systems. In the 15th Century, Leonardo recognized that the imposition of a force external to an object causes a change in the object. This led to the definition of **work** done on a system by an external force as the accumulation (integral) of force and displacement, compatible with the notion of a change in **potential energy, PE** if (potentially multicomponent) force is related to potential energy through a (potentially multicomponent) derivative, $f_i = -\dfrac{\partial PE(\mathbf{r})}{\partial r_i}$:

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[11] Recall that $\sin(0) = 0$ and $\cos(0) = 1$, so $f(0) = B$ and $\dot{f}(0) = A$.

$$\boxed{\begin{aligned} \mathbf{f} &= -\nabla \mathrm{PE}(\mathbf{r}) && (5.9a) \\ \mathrm{PE}(\mathbf{r}) &= -\int \mathbf{f}(\mathbf{r}) \cdot d\mathbf{r} && (5.9b) \end{aligned}}$$

$$\boxed{w_{PE} \equiv \int_i^f \mathbf{f}(\mathbf{r}) \cdot d\mathbf{r} = -\mathrm{PE}_f - (-\mathrm{PE}_i) = -\Delta \mathrm{PE} \qquad (5.10)}$$

Leibnitz deduced in 1686 that when work results in motion, there is a change in another quantity called **kinetic energy, KE**. From Newton's law, $\mathbf{f} = m\mathbf{a} \equiv \frac{\Delta \mathbf{v}}{\Delta t}$ and $d\mathbf{r} = \mathbf{v}dt$

$$\boxed{\mathrm{KE} \equiv \frac{mv^2}{2} = \frac{p^2}{2m} \qquad (5.11)}$$

$$\boxed{w_{KE} \equiv \int_i^f \mathbf{f} \cdot d\mathbf{r} = \int_i^f m \frac{d\mathbf{v}}{dt} \mathbf{v}dt = \frac{mv_f^2}{2} - \frac{mv_i^2}{2} = \mathrm{KE}_f - \mathrm{KE}_i = \Delta \mathrm{PE} \qquad (5.12)}$$

Kinetic energy is due to motion of objects while potential energy is due to interactions between objects.[12] The universal law which relates kinetic energy to potential energy is that *the sum of the potential and kinetic energies, or total energy, of any isolated system is a constant (conserved).* Equating the two expressions for work in terms of energy leads to the law of conservation of energy.

$$W_{KE} = w_{PE}$$

[12] Alternative notation is T for KE, V for PE, and H for E. We reserve T for temperature and V for volume.

$$KE_f - KE_i = PE_i - PE_f$$

$$KE_i + PE_i = KE_f + PE_f = E_{total}$$

Since the total energy is the same for the final and initial states, it remains constant (conserved).

$$\boxed{E_{total} \equiv PE + KE = \text{constant} \qquad (5.13)}$$

This is the *Law of conservation of energy* which governs all known objects of all kinds. From this law the description of the state and motion of objects may be determined.

While kinetic energy is absolute (zero at zero velocity), potential energy is *relative* in the sense that it measures attractions and repulsions and is characteristic of the type of interaction, including the gravitational attractions between all material objects (having mass), and electrostatic attractions and repulsions between charged objects (opposite charges attract and like charges repel). As the strength of interactions (usually) depends on distance, potential energy is a function of separation. The magnitude of the energy of a system must be defined in terms of some defined zero. In the case of gravitational or electrostatic interactions, it is convenient to define the zero of energy to be at infinite separation, in the case of vibrating systems, the zero of energy occurs at equilibrium (rest). The only truly absolute energy would be the total energy of the universe, an obviously difficult quantity to determine.

Gravitational and electrostatic potential energy fall off inversely with distance, while that of a spring (oscillator) increases with stretching. The functional form of potential energy is derived from that of force using Eq. (5.9b). For Eqs. (5.5) through (5.7)

$$PE_{gravitational}(r) = \frac{k_g}{r}, \qquad k_g \equiv -Gm_1m_2 \qquad (5.14a)$$

$$PE_{electrical}(r) = \frac{k_e}{r}, \qquad k_e \equiv q_1q_2 \qquad (5.14b)$$

$$PE_{oscillation}(r) = k_o r^2, \qquad k_o \equiv m\omega^2 \qquad (5.14c)$$

The proportionality constants contain the parameters (constant for a given system) for each type of system.

Fig. 5.1 compares these potential energy forms.

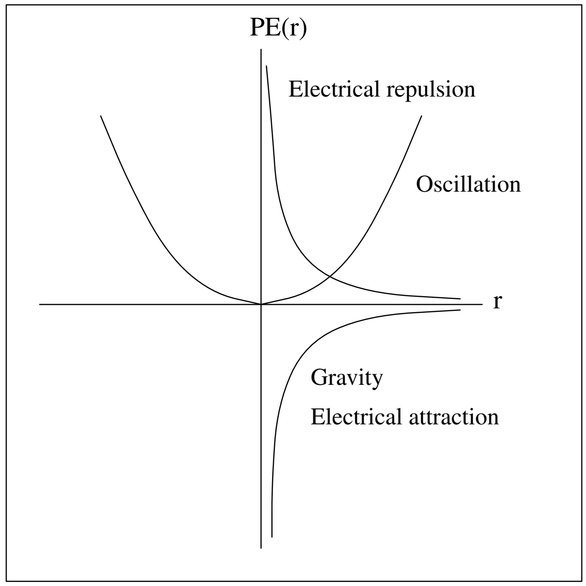

Fig. 5.1 Potential Energy for Typical Systems

Example 5.6 The change in gravitational potential energy of an object of mass m changing its altitude by Δh near the surface of the earth of radius $R_E >> \Delta h$ can be derived from the general expression Eq. (5.14a):

$$\Delta PE = -GmM_E\left(\frac{1}{R_f} - \frac{1}{R_i}\right) = -GmM_E\left(\frac{R_i - R_f}{R_i R_f}\right) = GmM_E\left(\frac{\Delta h}{R_E(R_E + \Delta h)}\right) \approx mg\Delta h, \qquad (5.15)$$

where $g \equiv GM_E/R_E^2$ is the gravitational constant on Earth, 9.80697 m/s^2 using the data of Example 5.3. This relation expressing the work needed to raise an object in the gravitational fi eld was fi rst mentioned by L.M.N. Carnot in 1803.

Example 5.7 A simple example of a child's swing illustrates energy concepts.

Fig. 5.2 shows the potential energy of an oscillating system. At rest the swing is at minimum potential (gravitational) energy, set arbitrarily to zero. If it is raised (by some input energy source) to a higher point it acquires additional potential energy. In the gravitational field the all the energy is potential energy with the value mgΔh according to Eq. (5.15). When released, the swing falls as gravitational potential energy is converted into kinetic energy. In motion, the maximum kinetic energy is realized at the bottom of the course of the swing where all the potential energy available is converted into kinetic energy (further fall is constrained by the rope). Conversely, at the top of the course of the swing all the kinetic energy is converted back into potential energy. The proof is that the velocity must pass through zero as it changes direction, and the kinetic energy becomes zero according to Eq. (5.11). At any point along the course of the swing the sum of the potential and kinetic energies is a constant (Eq. (5.13)). If no energy is lost from the system, the swing continues its motion forever. This is an ideal situation which assumes the swing is isolated, and ignores any interactions with its environment, such as connections to the support, atmosphere, etc. These interactions in a real case are called *friction* and dissipate energy of the swing to its surroundings, causing the motion of the swing to eventually cease. However, even in the real situation the total energy of the larger system is conserved.

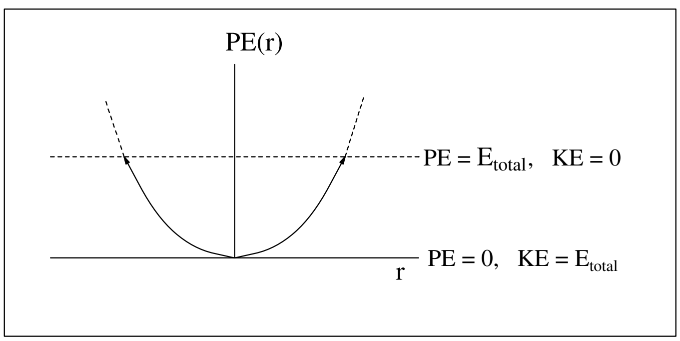

Fig. 5.2 Energy of an Oscillating System

It is possible to determine the maximum velocity of the swinging object from the law of conservation of energy. At the top of the swing trajectory,

$$E_{total} = PE = mg\Delta h$$

while at the bottom

$$E_{total} = KE = \frac{1}{2} mv_{max}^2$$

Equating the two expressions for the total energy and solving for velocity gives

$$v_{max} = \sqrt{2g\Delta h}$$

The higher the swing starts out, the greater the velocity at the bottom (but not proportionately). The fact that the motion of a swinging object is independent of the length of the connection to the support was first noticed by Galileo.[13]

Transforming from force to energy provides certain advantages to the analysis of mechanical systems. Mathematicians in the Nineteenth Century transformed Newton's law, which is a set of second-order differential force equations in N variables, into a set of 2N first-order differential energy equations. The total energy is symbolized by H and called the *hamiltonian* of the system, in honor of Sir William Rowan Hamilton who developed the energy form of mechanics in the 1830's. The space of variables is generalized displacements $q \equiv \{q_i\}$ and corresponding momenta $p \equiv \{p_i\}$ where p and q are shorthand representations standing for all their components. The q could be Cartesian coordinates, polar coordinates, or any others convenient to the description of the system. The set of displacement and momentum variables is called *phase space*.

For any function f(x) of a set of variables $x \equiv \{x_i\}$ which are themselves functions of another variable x(y), by the chain rule of differentiation (Section 3.11), the total derivative of f is the sum of an implicit derivative and an explicit derivative:

$$\frac{df}{dy} = \sum_i \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial y} + \frac{\partial f}{\partial y} \tag{5.16}$$

[13] Supposedly from observing the motion of swinging chandeliers in a cathedral during mass.

The partial derivatives denoted by the symbol ∂ simply means all variables other than those with which the derivative is being taken (e.g. x_i) are held constant. Since the total energy of a system is in general a function of all the variables of phase space as well as the time, H(p,q,t). According to the chain rule the total derivative of energy with respect to time is

$$\boxed{\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial q_i}\dot{q}_i\right) + \frac{\partial H}{\partial t} \qquad (5.17)}$$

where the dot denotes time derivative (Newton's notation). For systems for which the energy is conserved (constant in time) $\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$, H = H(p,q), and Eq. (5.17) is satisfied if the summation terms are each zero as well. Since $\dot{q}_i\dot{p}_i - \dot{p}_i\dot{q}_i \equiv 0$ this will be ensured for

$$\dot{q}_i = +\frac{\partial H}{\partial p_i} \qquad (5.18a)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (5.18b)$$

From the symmetry of the equations is is easy to see why p_i and q_i are called *conjugate variables*. Eqs. (5.18) are 2N first-order differential equations called *Hamilton's equations*. *Hamilton's equations are equivalent to Newton's N second-order equations Eq. (5.4), and form an alternate approach to mechanics in terms of potential energy instead of force.* Thus for $H(p, q) = KE(p) + PE(q) = \frac{p^2}{2m} + PE(q)$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{\partial KE(p)}{\partial p_i} = \frac{\partial(\frac{p^2}{2m})}{\partial p_i} = \frac{p}{m}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = -\frac{\partial PE(q)}{\partial q_i} = f$$

The first equation is the definition of momentum, Eq. (5.3), while the second is Newton's law of force by Eqs. (5.4). The second equation also explains Eq. (5.9).

Example 5.8 Analyze the motion of a mass attached to a spring which is described by Hooke's Law in energy form.

There is only one coordinate, the displacement, which we will identify by q. According to Eqn. (5.14c) the hamiltonian is then $H(p,q) = KE(p) + PE(q) = \frac{p^2}{2m} + \frac{kq^2}{2}$. Hamilton's equations of motion of the spring, Eqs. (5.18), become

$$\dot{q} = +\frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -kq$$

Taking a second derivative of the first equation, we have

$$\ddot{q} = \frac{\dot{p}}{m} = -\frac{k}{m}q$$

which is identical to Newton's equation for the spring given in Example 5.5 and therefore has the same solution.

Since the last example ends up with the same mathematical equations as Newton's approach to mechanics, one might wonder what the advantage of Hamilton's approach is. Hamilton's equations show directly that any variable which does not appear in the hamiltonian will lead immediately to a conservation law for that variable since according to Eq. (5.18b) $\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0 \Rightarrow p = \text{constant}$. Conservation is a very fundamental property of a system having to do with symmetry. Symmetry considerations permit immediate separation of a complex problem into simpler problems. This will be illustrated in the next section.

5.4. N-Body Mechanics

Thus far we have considered the interactions of two bodies (particles). The analysis of the motion of more than two interacting bodies does not admit mathematical solutions in closed form for the general case. Then how does one treat even a three-body problem, such as the earth-sun-moon motion? It is necessary to resort to numerical solutions for which the electronic computer was developed in the Twentieth Century. However it is always possible to separate a multiple-body problem into internal and external parts, which can be treated separately. This leads to the concepts of reduced mass and center of mass. We will develop the equations of motion for a two-body system to illustrate how motion separates.

If the interaction potential between two bodies depends only on the distance between them, it proves beneficial to transform from Cartesian coordinates to *center of mass* coordinates. Given

$$r_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

Let {X,Y,Z} be center of mass coordinates and {x,y,z} be internal coordinates defined by:

$$X \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \qquad Y \equiv \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \qquad Z \equiv \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}$$

$$x \equiv x_2 - x_1 \qquad y \equiv y_2 - y_1 \qquad z \equiv z_2 - z_1$$

The kinetic energy is then given by

$$KE = \frac{1}{2}(m_1 + m_2)(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

The quantity $M \equiv m_1 + m_2$ is the total mass of the system and quantity $\mu \equiv m_1 m_2/(m_1 + m_2)$ is called the *reduced mass* of the system. The total energy of the two-body system is

$$E = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + \frac{\mu}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - PE(x, y, z)$$

Since the center of mass coordinate (X,Y,Z) dependency is separated from the internal coordinate (x,y,z) motion, Hamilton's equations of motion lead to

$$M\ddot{X} = M\ddot{Y} = M\ddot{Z} = 0$$

$$\mu\ddot{x} = -\frac{\partial PE}{\partial x} \qquad \mu\ddot{y} = -\frac{\partial PE}{\partial y} \qquad \mu\ddot{z} = -\frac{\partial PE}{\partial z}$$

The first equation describes the motion of a body of mass equal to the total mass of the system which responds to any external forces (none in the case of an isolated two-body system). Interaction potentials which depend only on separation and not orientation suggest transforming to coordinates which reflect the symmetry of the system. If the internal coordinates are transformed to spherical polar coordinates a further separation of motion results.

$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$

The internal motion is determined by the internal Hamiltonian

$$E = \frac{\mu}{2}\dot{r}^2 + \frac{\mu}{2}r^2\dot{\theta}^2 + \frac{\mu}{2}r^2\sin^2(\theta)\dot{\phi}^2 + PE(r)$$

We need to express the Hamiltonian in coordinates and generalized momenta p, where $p \equiv \frac{\partial KE}{\partial \dot{q}}$ so that

$$p_r = \mu\dot{r}$$
$$p_\theta = \mu r^2\dot{\theta}$$
$$p_\phi = \mu r^2\sin^2(\theta)\dot{\phi}$$

The internal motion Hamiltonian then is

$$E = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + \frac{p_\phi^2}{2\mu r^2\sin^2(\theta)} + PE(r) \tag{5.19}$$

Hamilton's equations of motion for momenta (5.18b) then yield

$$\dot{p}_r = \frac{p_\theta^2}{\mu r^3} + \frac{p_\phi^2}{\mu r^3\sin^2(\theta)} - \frac{\partial PE(r)}{\partial r}$$
$$\dot{p}_\theta = \frac{p_\phi^2\cos(\theta)}{\mu r^2\sin^3(\theta)}$$
$$\dot{p}_\phi = 0$$

The third equation states that the angular motion about the z axis is conserved (constant) and therefore the motion must lie in a plane perpendicular to the z axis with $\phi = \pi/2$. The motion thus reduces to that in a plane described by plane polar coordinates r and θ. The first equation determines the internal motion according to the form of PE(r). Note that viewed as as a Newtonian force equation (cf. Eq. (5.4)), there is an additional force due to rotation (Corliolis force). If r is constant, $p_r = 0$ and the first equation of motion describes pure rotation of a system with a rotational "mass" equal to the moment of inertia $I \equiv \mu r^2$ and angular momentum squared $p^2_{angular} \equiv p^2_\theta + \frac{p^2_\phi}{\sin^2(\theta)}$. Hamilton's equations of motion for coordinates (5.18a) can be invoked to show that the total angular momentum is constant (conserved), even in the general case where r is not constant. Since the angular motion is conserved the internal motion is determined by the interactive potential. This is what has been assumed for the two-body gravitational, electrostatic and elastic potentials in the previous sections.

It is instructive to apply the equations of motion to two bodies connected by a potential such as that describing a spring, as this is a classical model for a diatomic molecule. In the general case, the molecule undergoes both rotation and vibration and is called a rot-vibrator. The effect of rotation is to add a rotational "barrier" to the quadratic potential and the first equation of motion has the form $\dot{p}_r = -\,\partial V(r)_{effective}/\partial r$ with

$$V(r)_{effective} = V(r) + \frac{p^2_{angular}}{2I(r)} \tag{5.20}$$

To a first approximation (where I is assumed to be constant) the total energy is the sum of two terms, one describing the vibration and the other describing the rotation.

5.5. Field Mechanics

Gravitational, electrical and magnetic interactions were observed anciently. These forces acting over distance suggest some connection between objects over space. This spatial extension of influence is called a **field**. Light does not appear at first to be related to mechanical and electrical forces, but does share with them the ability to cause change. We might say they all contain or convey energy. The discovery in the Nineteenth Century that electricity, magnetism and light could be unified into one model generated a quest for a unified field theory for all interactions in the twentieth.

Newton made equally profound contributions to the understanding of light as he did to material objects. Newton used both a "corpuscular"[14] or particle model to explain some aspects of the behavior of light, such as shadows (linear motion), reflection (returning from

[14] A term first coined by Robert Boyle in 1666 to describe the atomic particles of matter.

an interface) and refraction (bending through an interface), and a wave model of light to explain others, such as interference (regions of increased and decreased intensity of merging beams) and color. Using a simple prism he demonstrated that light from the sun is not one color but rather consists of a dispersion or "spectrum" of colors as seen in rainbows. The various colors were associated with characteristic wavelengths of a periodic disturbance of a transmitting medium, called the *luminiferous aether*. Newton correctly determined the wavelengths of visible light to be of the order of fractions of a micrometer (millioneth of a meter). The **wavelength**, λ (Greek lambda) is the distance between recurring peaks of intensity along the direction of propagation; the **frequency**, ν (Greek nu, pronounced "new") measures the rate of repetition to a stationary observer as the waves pass by. Frequency and wavelength are related through the velocity of the disturbance, given the symbol c in the case of light:[15]

$$\boxed{\lambda\nu = \mathrm{c} \qquad (5.21)}$$

Because there is an *inverse* relationship between wavelength and frequency, the proportionality factor c acts as a conversion factor as depicted in Fig. 5.3.

[15] The velocity of light (in a vacuum), c = 299,792,458 m/s.

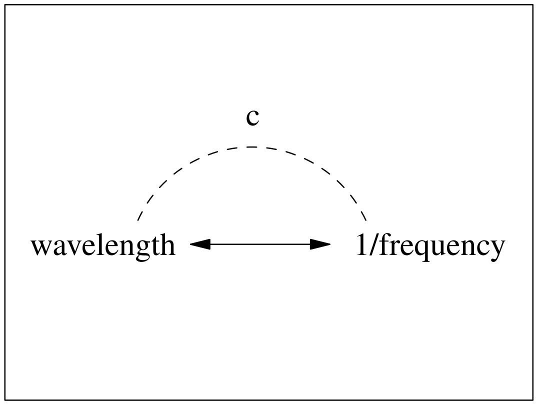

Fig. 5.3 The Wavelength/Frequency Conversion Map

Example 5.9 What frequency of light corresponds to red light of wavelength 700 nm?

By Eq. (5.21)

$$700 \text{ nm } \left(\frac{1 \text{ m}}{10^9 \text{ nm}}\right)\left(\frac{1 \text{ sec}}{2.998 \text{ x } 10^8 \text{ m}}\right) = 2.335 \text{ x } 10^{-15} \text{ s} = \frac{1}{\nu}$$

Therefore, $\nu = 4.28 \text{ x } 10^{14} \text{ s}^{-1}$.

In the Nineteenth and Twentieth Centuries the wave model of light was extended to regions beyond the visible spectrum to wavelengths longer than those of red light, or *infrared* light (corresponding to thermal, or heat radiation), having wavelengths up to a millimeter, and to shorter than violet, or *ultraviolet* light, with wavelengths ranging down to nanometers.

In 1856 James Clerk Maxwell[16] developed an unified theory of electricity, magnetism and light based on a model of radiating ‘‘electromagnetic’’ waves generated by the acceleration motions of oscillating charges.[17] The discussion of wave models in Chapter 4 describes

[17] James Clerk Maxwell (Scottish, 1831-1879) ranked with Newton as a mathematical physicist, published his first mathematical paper at age 14, won the Adams Prize at age 25 for proving that the rings of Saturn could not remain solid under their mutual forces, developed unified theories of heat and of electricity, magnetism, and

some of the mathematical relationships for waves. Maxwell's electromagnetic spectrum eventually included microwave and radio waves having wavelengths ranging between centimeters to kilometers, and x-rays and gamma rays with wavelengths down to picometers and femtometers, respectively. Waves tend to bend around obstacles producing *diffraction* patterns of alternating intensity with spacing equal to the wavelength and falling off with distance from the obstacle. Radio waves are long enough to be detected behind sizable obstacles, but microwaves used for television transmission require direct visibility to the source for good detection. Visible light diffraction patterns are too small to have been detected in Newton's time although interference patterns from thin films such as soap bubbles could be seen.

A profound consequence of Maxwell's model was that the velocity of propagation of electromagnetic disturbances was both finite and absolute. Maxwell's model is a theory of fields of influence, distinct from particle models. It stimulated the development of two Twentieth Century refinements of mechanics, quantum mechanics and relativistic mechanics. It relied on the existence of a medium (the ether) which had the conflicting properties of conveying disturbances such as light with enormous velocities yet allowing material objects to pass insensibly through it. Thus ether was some kind of medium apparently having simultaneously very high as well as very low density. The resolution of this dilema and the exploration of an absolute value for the velocity of electromagnetic radiation led us into the revolutions of modern physics.

5.6. Relativistic Mechanics

In Newtonian mechanics, space and time provide a framework for the motion of particles. Their supposed independence from each other and from objects was reevaluated near the beginning of the Twentieth Century by Albert Einstein[18], who generalized Newtonian

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light, succumbed to cancer at the same age of his mother, 48.

[17] Light was added to the electromagnetic theory upon discovery that the speed of propagation of electromagnetic waves (equal to the ratio of the electromagnetic to electrostatic units of charge) is the same as the measured speed of light.

[18] Albert Einstein (German, 1879-1955) considered with Isaac Newton as the greatest physicists of all time. Published three seminal papers in 1905 on special relativity, Brownian motion and the photoelectric effect. Awarded the Nobel Prize in physics in 1921 for the latter, as relativity was not yet verified. There was only one book found in Einstein's office after his death, and that had been left by a previous occupant.

Among his less famous numerous 1905 publications is his doctoral dissertation, titled (in English translamechanics to relativistic mechanics. The basic notions of relativistic mechanics are rooted in common experience, such as the effects of elevators on objects. At rest or moving at constant velocity, Newton's Law of gravitation applies. Acceleration downward at the value of the acceleration of gravity cancels the effect of gravity, whereas acceleration upward at the same value doubles the gravity force. Thus acceleration isolated in space produces the same effects as gravity. Einstein's contribution was to explain the equivalence of inertial and gravitational mass, called the *principle of equivalence*.[19]

The object of relativistic mechanics is to cast the laws (equations) of physics into forms which are are independent of the motion of the frame of reference (same or "covariant" form).[20] For frames of reference which move at constant relative velocity (Galilean frames), this is achieved through recognition of the equivalence of space and time (the "fourth dimension"). For frames of reference in relative acceleration, this is achieved through recognition of the curvature of space/time due to motion or matter. Universal law (independence of frames of reference) was *assumed* in Newtonian mechanics, but the assumption of the universality of the speed of light in a vacuum from electromagnetic mechanics suggested the erroneous notion that absolute motion could be detected.[21] However, all attempts to detect absolute motion of say, the earth through space (or the ether) were doomed to failure, as Einstein showed in 1905.

Consider a falling object in a frame of reference which is in constant motion with respect to a second frame of reference, which in turn may be considered stationary with

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tion) *A New Determination of Molecular Dimensions*.

[19] Weightlessness in an orbiting space station and its rotation to produce "artificial" gravity are applications of the principle of equivalence.

[20] "Motion is like nothing." Galileo, 1638. "The laws of physics must be of such a nature that they apply to systems of reference in any kind of motion.", A. Einstein, 1916.

[21] Conceptually, Newton's equations of particle mechanics are relativistic in the sense that f = ma involves acceleration, or change in velocity. Hence objects moving with respect to each other at constant velocity experience the same laws of motion. Hence apples accelerate toward the earth the same way the moon does to keep it from flying out of its orbit; thus comets, solar systems, stars and galaxies obey the same laws of gravitation.

The unification of electric and magnetic phenomena through Maxwell's equations of electromagnetic field mechanics (not given here: see a text on electricity and magnetism) involve velocity and are only relativistic in the sense that a moving charge (changing electric field) generates a magnetic field (Ampere's Law) and a moving magnet (changing magnetic field) generates an electric field (Faraday's Law).

respect to the moving frame. In the moving frame it appears to fall vertically a certain distance in a given time, but in the stationary frame, the motion of the moving frame causes it to appear to fall diagonally for a longer distance in the same time (cf. Fig 5.4). If however, the object is a beam of light, according to the postulate that the speed of light is an universal constant in all frames of reference, the object must take a longer time to fall the further distance in the stationary frame than in the moving frame. The *time dilation* factor may be computed from an application of the Pythagorean theorem to the situation as depicted in Fig. 5.4, in which primed quantities refer to the moving frame and unprimed quantities refer to the stationary frame, and the velocity of light, c, has the same value in both frames.

$$(ct)^2 = (ct')^2 + (vt)^2$$

$$1 = \left(\frac{t'}{t}\right)^2 + \left(\frac{v}{c}\right)^2$$

$$\frac{t}{t'} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{5.22}$$

Of course, from the vantage point of the moving frame, the stationary frame appears to be moving, and hence appears in the moving frame to have a time dilation. Each observer would claim that the other's clock is running slow!

Note that time dilation refers to *intervals* of time. Relativity makes a similar statement about intervals of distance: lengths appear to be contracted in frames moving relative to an observer's frame. The relative velocity (which is common to both systems) refers to distance traveled by the moving system during a time interval measured in the moving system.

$$v = \frac{x}{t'} = \frac{x'}{t}$$

Hence

$$\frac{x}{x'} = \frac{t'}{t} = \sqrt{1 - \left(\frac{v}{c}\right)^2} \tag{5.23}$$

which is less than unity for $0 < v < c$.

Thus as velocity approaches the speed of light, mass increases, time slows down and space contracts.

Relativity asserts the impossibility to detect absolute motion. Detection connotes transmission of information. The pitch of a sound generated on a moving vehicle (train, automobile, etc) changes as it approaches and recedes from a stationary listener. This is due to the fact that sound information is conveyed by colliding molecules whose velocities are additive with that of the source. Not so with electromagnetic radiation which conveys information at a universal speed independent of the motion of the source. Thus, two observers in relative motion should both observe information radiating outward at velocity c in a sphere of the form

$$x^2 + y^2 + z^2 = c^2t^2$$

Einstein applied the equations of special relativity to Newtonian mechanics and was led to correction terms to the kinetic energy of particles.

$$\boxed{\Delta E = c^2 \Delta m \qquad (5.24)}$$

Δ (delta) refers to change, as usual, from initial state to final state. This result from special relativity states that there is a proportionality between a change in mass, Δm, and change in energy with the proportionality constant equal to the square of the velocity of light, c.

Consideration of invariance of equations of motion in frames in relative acceleration led Einstein to the theory of general relativity in 1916. Acceleration or equivalently mass warps or curves space-time coordinates. Four consequences of general relativity have supporting experimental evidence: a slight bending of light passing by massive celestial objects, a slight slipping (advance of perihelion) of the orbit of planets, a slight shift in frequency of light near massive celestial objects, and the expansion of the universe. While of cosmological significance, these effects are minimal for non-gravitational systems such as atoms and molecules.

5.7. Quantum Mechanics

Experiments in 1887 by Heinrich Hertz confirmed Maxwell's speculation that electromagnetic radiation of all types (including light and heat) traveled with the same speed. A bonus was the discovery of that light could stimulate the generation of an electric current, or flow of electrons through space from active metals (like cesium), called the *photoelectric* *effect*. Exploration of the photoelectric effect showed that the strength of current, or number of electrons generated over time, is proportional to the strength or intensity of light. But the kinetic energy of the electrons is independent of intensity of light and depends instead on the frequency of the light. Electrons are produced above a threshold frequency characteristic of the active metal with varying kinetic energies up to a maximum value which is proportional to the frequency of the stimulating light. The proportionality constant was found to have the same value as that discovered by Max Planck[22] in 1901 in his *quantum mechanical*[23] explanation of the distribution of heat radiation. According to Maxwell's electromagnetic theory, glowing bodies like the sun radiate a spectrum of light and heat frequencies. After identifying a function which describes the energy distribution of glowing bodies empirically,[24] Heat radiation is based on a model of vibrating molecules which emit heat energy by changing vibrational energy states. Planck discovered that the experimental energy distribution could be explained only by restricting the molecular energy states to discrete values.[25] In revolutionary contrast to classical mechanics, which would permit a continuous distribution of energy states, Planck found that vibrational energies were restricted to the values

$$\boxed{E_n = nh\nu \qquad (5.25)}$$

where ν is the frequency of vibration, n is a positive integer, called a *quantum number*, and h is a constant, now called Planck's constant, with value 6.626076×10^{-34} J s. Light emitted at

[23] Max Karl Ernst Ludwig Planck (German, 1858-1947) already had an established career as a thermodynamicist when he discovered an empirical equation that describes the spectrum of hot glowing bodies. Over the winter vacation of 1900 he spent the most arduous weeks of his life deducing that the equation required discrete or quantized motion of the oscillator model he used to describe the generation of the light spectrum. When Einstein announced through independent analysis in 1905 that *space* was quantized, Planck retracted the quantum hypothesis. In 1913 he wrote: "That [Einstein] may sometimes have missed the target in his speculations, as, for example, in his theory of light quanta, cannot really be held against him." After being awarded the Nobel Prize in physics in 1918, Planck relented. One of his Planck's sons was executed by the Nazis for attempting to assassinate Hitler.

[23] Latin: *quantus*, how much, *mechanicus*, machine.

[24] That is, trying educated guesses.

[25] The restriction resulted from retaining a *finite* integration interval (dx), rather than allowing the Newtonian calculus limiting value of zero.

frequency ν is due to a transition between adjacent energy states:

$$\boxed{\Delta E = E_{n+1} - E_n = h\nu \qquad (5.26)}$$

The reason radiation appears continuous to the most sensitive detectors is because of the small magnitude of Planck's constant.

In 1905 Einstein identified the kinetic energies of photoelectrons (electrons ejected from active metals by ultraviolet light) with ΔE of Eq. (5.26). Einstein's contribution[26] was to suggest that the radiation field (that is, space), or light itself is discrete (quantized). Einstein's "hunks" electromagnetic energy were later dubbed light *photons*[27] by G. N. Lewis. Eqs. (5.25) and (5.26), which may be referred to as the Einstein-Planck equations, reflect a dual nature of light. One side describes a discrete energy (photon) form and the other a field-based frequency (wave) form.[28]

5.8. Wave Mechanics

Wave mechanics was developed from a synthesis of field and quantum mechanics by Erwin Schrödinger[29] in 1925. We will sketch how the quantum mechanical wave equation is related to classical wave phenomena. Although the proper context is advanced mathematics,

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26 Apparently totally independent of Planck's, as indicated by the fact that Einstein's publication makes only a passing reference to Planck's work as if an afterthought, perhaps suggested by a reviewer.

27 Greek: *photos for light* + *on* for stuff.

28 A simple observation demonstrates the *particle* nature of light; starlight is so dim that according to classical field mechanics if light were solely a wave phenomenon, it would take more than a lifetime for the eye to absorb enough energy to detect the light from a star, yet bundles of energy delivered by particles of light may be detected instantly. Another simple observation demonstrates the *wave* nature of light; squinting at a point source of light produces interference patterns of alternating light and dark regions as the light passes through alternate paths around the eyelashes, characteristic of the interaction of waves.

Note that neither a wave nor particle model of existence (or some combination) can be complete since each requires spatial extension, hence some form of substructure. Without attempting to explain any apparent contradictory nature of duality we will simply accept it as an hypothesis.

29 Erwin Schrödinger (Austrian, 1887-1961) Nobel Prize in physics, 1933. Wrote *What is Life* in 1944, explaining the roles of statistics, uncertainty, entropy and stars in biology. Developed wave mechanics following a suggestion of Einstein.

we will simplify the discussion to limit the mathematics to algebra as much as possible. This will not be an actual derivation because the wave equation is now considered a fundamental law of nature, the same way that Newton's Second Law of mechanics, F = ma, is fundamental to classical mechanics. In fact, the wave equation is more fundamental than Newton's equation since it gives the same results as classical mechanics for macroscopic objects, yet is capable of describing the behavior of the smallest particles, which classical mechanics is not. Before we can discuss the quantum aspects of waves, we must first understand something about waves in general.

The first step in making the transition from classical wave phenomena to quantum wave phenomena uses the Planck-Einstein equation (Eq. 5.26) and the Einstein special relativity proportionality between energy and mass, $\Delta E = mc^2$, to relate the wavelength of light to its velocity:

$$\Delta E = h\nu = \frac{hc}{\lambda} = mc^2 \qquad (5.27)$$

Louis DeBroglie[30] made the bold leap of generalizing the last equality *to include matter as well* (replacing c with v):

$$mv\lambda = h \qquad (5.28)$$

Solving this equation, called the *de Broglie equation*, for velocity and Substituting into the definition of kinetic energy, KE, relates kinetic energy to wavelength:

$$KE = \frac{1}{2}\ mv^2 = \frac{m}{2}\ (\frac{h}{m\lambda})^2 = \frac{h^2}{2m\lambda^2} \qquad (5.29)$$

The connection to the wave equation, Eq. 4.12, is made through λ: solve Eq. 4.12 for λ and substitute into Eq. (5.29).

$$KE = -\ \frac{h^2\nabla^2\psi}{8\pi^2 m\psi} \qquad (5.30)$$

Using the fact that the sum of the kinetic and potential energy, PE, equals the total energy, E, we have (multiplying through by ψ to clear it from the denominator, and factoring ψ out on

[30] Louis Victor Pierre Raymon Duc de Broglie (French, 1892-1977) an influential prince and scientist who promoted duality. Awarded the Nobel Prize in physics in 1929.

the left hand side)

$$\boxed{(-\frac{h^2}{8\pi^2 m}\nabla^2 + PE)\psi = E\psi, \qquad (5.31)}$$

which is *Schrödinger's quantum mechanical wave equation*. Various systems are distinguished by the forms of their potential energy. Schrödinger's equation, being a differential equation, yields wave functions (eigenfunctions) and constants (eigenvalues) for its solutions.

The *algorithm* for solving Schrödinger's equation is easy to state: insert the form of the potential energy into the equation and solve for the eigenfunctions. Introducing boundary constraints on the eigenfunctions generates eigenvalues just as for classical wave motion. For a simple one-dimensional harmonic oscillator (like a guitar string), the potential energy follows Hooke's Law, $PE = kr^2$, corresponding to a restoring force proportional to the displacement. The interactions in atoms an molecules are principally electronic attractions and repulsions, which obey Coulombs Law, $PE = \sum \frac{q_i q_j}{r_{ij}}$, where the sum is over all pairs of charged particles (electrons and nuclei) with charges q_i and q_j separated by distance r_{ij}.

Although it is fairly simple to write Schrödinger's equation for an atom or molecule, solving it is another matter. Solutions in terms of known functions are not available for any but the simplest systems, and it must be solved numerically. Programs to implement the numerical solution to Schrödinger's equation involve literally tens of thousands of lines of code and tax the most sophisticated super computers for anything but the simplest molecules. It is indeed fortunate that simplifying approximations to Schrödinger's equation lead to meaningful results for complicated systems.

5.9. QED, etc

Einstein sought to unify gravitation with electromagnetism as Maxwell had unified electricity and magnetism, but didn't succeed. New forces, strong and weak nuclear forces have been added to the list and the search for a ''grand unified filed theory'' ɸ ''theory of everything'' continues. The synthesis of quantum mechanics and electromechanics, called quantum electrodynamics, or QED, has proven useful in explaining a wide variety of physics phenomena.

Summary

The foundation for physics is mathematics. The first field of mathematical physics is mechanics, which describes the state and motion of physical bodies. The measure of change is force and energy. Equations of motion for force are second-order differential equations, while equations of motion for energy are first-order differential equations. Kinetic energy is an universal form for all systems (dependent on the coordinate system). Systems are distinguished by the forms of their interaction potential energies. Physics has evolved from descriptions of relatively slow macroscopic bodies to descriptions of fast microscopic bodies, using relativistic and quantum mechanics, respectively. Field mechanics describes wave phenomena and a synthesis of all mechanics is the major goal of physics.

MECHANICS EXERCISES

1. On July 6, 1994 Leroy Burrell ran 100 m in 9.85 s. What was his average speed in *mi/hr*?
2. What is the final speed of a dragster race car which travels 1/4 mi in 8.0 s from a standing start?
3. How much kinetic energy in kJ has a 150 lb person running 25 mi/hr, and how far would they have to fall in the Earth's gravitational field to acquire that amount of energy?
4. What is the form of the potential energy for an harmonic oscillator?
5. What is the speed at the bottom of the swing of a child who has been lifted 2 meters and released?
6. If energy is conserved, how can a person on a swing "pump" themselves higher?

MECHANICS EXERCISE HINTS

1. Consider the definition.
2. Assuming constant acceleration, it is possible to show that average velocity, $\bar{v}$, is the average of the final and initial velocities:[31]

$$\bar{v} = \frac{v_f + v_i}{2}$$

[31] From the definition of the average of a function of one variable, the average of the velocity over time is:

$$\bar{v} \equiv \frac{\int_i^f v\,dt}{\int_i^f dt} = \frac{\int_i^f v \frac{dv}{a}}{\int_i^f dt} = \frac{\frac{1}{a}\left.\frac{v^2}{2}\right|_i^f}{\left.t\right|_i^f} = \frac{\frac{\Delta t}{\Delta v}\left(\frac{1}{2}\right)(v_f^2 - v_i^2)}{\Delta t} = \frac{1}{2}\,\frac{v_f^2 - v_i^2}{v_f - v_i} = \frac{1}{2}(v_f + v_i)$$

3. Newton could work this problem.
4. Harmonic oscillators follow Hooke's Law.
5. The result is independent of the mass of the child.
6. Consider the source.