SUPPLEMENT #0 -- CALCULUS TECHNIQUES

Differentiation: Multiply coefficient in front by exponent. Subtract one from the exponent.

Example: Differentiate $y = 3x^5$

$$3\,x^{5-1} \rightarrow 5 \cdot 3\,x^{4}$$

Answer: $y' = 15x^4$

Integration: Add one to the exponent. Divide coefficient in front by new exponent. Put '+C' on answer.

Example: $\int 15x^4\,dx$

$$\frac{15}{}\,x^{4+1} \rightarrow \frac{15}{5}\,x^{5}$$

Answer: $3x^5 + C$

Rules of Differentiation:

Sum Rule: Differentiate each term as if it were all by itself.

Example: $y = 3x^5 - \frac{x}{4} + \frac{4}{x} - 3\sqrt{x} - 6$

Step 1: Rewrite equation to make it easier to see differentials: $3x^5 \; - \frac{1}{4}x^1 \; + 4x^{-1} \; - 3x^{\frac{1}{2}} \; - 6$

(Recall: Negative exponents move to the other side of a fraction and become positive exponents.
Fractional exponents are radicals $\sqrt{\;}$)

Step 2: Differentiate each term separately (Multiply coefficient in front by exponent. Subtract one from the exponent):

$3x^5 \Rightarrow 15x^4$

$-\frac{1}{4}x^1 \Rightarrow 1\cdot\left(-\frac{1}{4}\right)x^{1-1} \Rightarrow -\frac{1}{4}x^0 \Rightarrow -\frac{1}{4}\cdot 1$ (Recall: anything to the zero power equals '1')

$+4x^{-1} \Rightarrow -1\cdot(+4)x^{-1-1} \Rightarrow -4x^{-2} \Rightarrow -\frac{4}{x^2}$

$-3x^{\frac{1}{2}} \Rightarrow {}^{1}/_{2}\cdot -3x^{\frac{1}{2}-1} \Rightarrow -\frac{3}{2}x^{-\frac{1}{2}} \Rightarrow -\frac{3}{2\sqrt{x}}$

$-6 \Rightarrow 0$ (Important concept: CONSTANTS ALWAYS DIFFERENTIATE TO ZERO!)

Step 3: Reassemble our newly differentiated equation:

$y' = 15x^4 - \frac{1}{4} - \frac{4}{x^2} - \frac{3}{2\sqrt{x}}$

Power Rule: OI! Differentiate the outside, then differentiate the inside, then MULTIPLY the two together

Example: $3(x^3 + 7x^2 - ½x + 3)^{5-1}$

Step 1: Differentiate the outside: $5\cdot 3(x^3 + 7x^2 - ½x + 3)^4 \Rightarrow 15(x^3 + 7x^2 - ½x + 3)^4$

Remember: Constants differentiate to ZERO!

Step 2: Differentiate the inside: $1x^3 + 7x^2 - ½x^1 + 3 \Rightarrow 3\cdot 1x^{3-1} + 2\cdot 7x^{2-1} - ½x^{1-1} + 0 \Rightarrow 3x^2 + 14x - ½$

Step 3: Multiply steps 1 & 2 together: $15(x^3 + 7x^2 - ½x + 3)^4(3x^2 + 14x - ½)$

This is the answer. Woo-Hoo!

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Example: $y = 3(\underbrace{x^3 + 7x^2 - \tfrac{1}{2}x + 3}_{u})^5$

Differentiate

1. **Multiply coefficient in front by exponent.**
2. **Subtract one from the exponent**

Step 1: Let $u = x^3 + 7x^2 - \tfrac{1}{2}x + 3$

Re-write the Example: $y = 3(u)^5 \;=\; 3u^{5-1}$

$\frac{dy}{du} = 5 \cdot 3u^4 \Longrightarrow 15u^4$

Remember: Constants differentiate to ZERO!

$\frac{du}{dx} = 1x^3 + 2x^2 - \tfrac{1}{2}x^1 + 3 \Longrightarrow 3 \cdot 1x^{3-1} + 2 \cdot 7x^{2-1} - \tfrac{1}{2}x^{1-1} + 0 \Longrightarrow 3x^2 + 14x - \tfrac{1}{2}$

Chain Rule: $\frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{15u^4} \cdot \underbrace{\frac{du}{dx}}_{3x^2 + 14x - \frac{1}{2}}$

Remember Step 1:
Let $u = x^3 + 7x^2 - \tfrac{1}{2}x$

Final Answer: $15(\underbrace{x^3 + 7x^2 - \tfrac{1}{2}x + 3}_{u})^4(3x^2 + 14x - \tfrac{1}{2})$

Product Rule: $u'v + uv'$

Example: $(x^2 - 1)^4(x^2 + 1)^5 \Longrightarrow \underbrace{(x^2 - 1)^4}_{u}\underbrace{(x^2 + 1)^5}_{v}$

$u = (x^2 - 1)^4$

$u' = \underbrace{4(x^2 - 1)^3}_{\text{Outer}}\underbrace{(2x)}_{\text{Inner}} \Longrightarrow 8x(x^2 - 1)^3$

$v = (x^2 + 1)^5$

$v' = \underbrace{5(x^2 + 1)^4}_{\text{Outer}}\underbrace{(2x)}_{\text{Inner}} \Longrightarrow 10x(x^2 + 1)^4$

Apply the product rule: $\underbrace{u'}_{8x(x^2 - 1)^3}\ \underbrace{v}_{(x^2 + 1)^5} + \underbrace{u}_{(x^2 - 1)^4}\ \underbrace{v'}_{10x(x^2 + 1)^4}$

Final Answer: $8x(x^2 - 1)^3(x^2 + 1)^5 + 10x(x^2 - 1)^4(x^2 + 1)^4$

Sometimes we want to get fancy and factor the final answer: $8x(x^2 - 1)^3(x^2 + 1)^5 + 10x(x^2 - 1)^4(x^2 + 1)^4$

$2x(x^2 - 1)^3(x^2 + 1)^4[4(x^2 + 1) + 5(x^2 - 1)] \Longrightarrow 2x(x^2 - 1)^3(x^2 + 1)^4[4x^2 + 4 + 5x^2 - 5]$

$2x(x^2 - 1)^3(x^2 + 1)^4\underbrace{(9x^2 - 1)}_{\text{Factors by difference of squares}} \Longrightarrow \boxed{2x(x^2 - 1)^3(x^2 + 1)^4(3x + 1)(3x - 1)}$

Quotient Rule: $\frac{u'v - uv'}{v^2}$

Example: $\frac{(x^2-1)^4}{(x^2+1)^5} \Rightarrow \frac{(x^2-1)^4}{(x^2+1)^5}$ } $\frac{u}{v}$

$u = (x^2 - 1)^4$

$u' = \underbrace{4(x^2 - 1)^3}_{\text{Outer}}\underbrace{(2x)}_{\text{Inner}} \Rightarrow 8x(x^2 - 1)^3$

$v = (x^2 + 1)^5$

$v' = \underbrace{5(x^2 + 1)^4}_{\text{Outer}}\underbrace{(2x)}_{\text{Inner}} \Rightarrow 10x(x^2 + 1)^4$

Apply the quotient rule: $\frac{\overbrace{8x(x^2-1)^3}^{u'}\ \overbrace{(x^2+1)^5}^{v} - \overbrace{(x^2-1)^4}^{u}\ \overbrace{10x(x^2+1)^4}^{v'}}{[(x^2+1)^5]^2}$ } V^2

When we take a power to a power, we MULTIPLY!

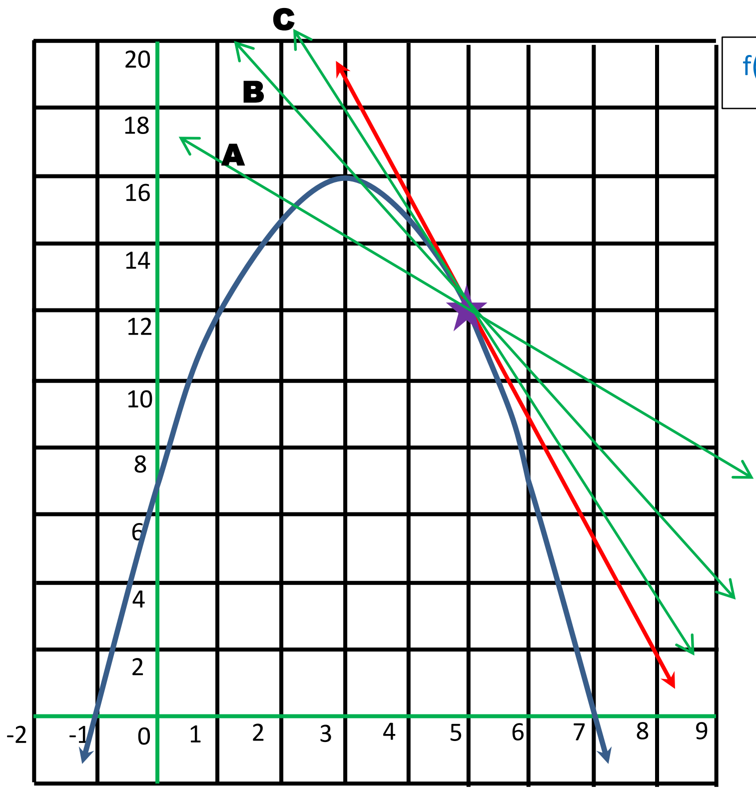
Factor Completely: $\frac{8x(x^2 - 1)^3(x^2 + 1)^5 - 10x(x^2 - 1)^4(x^2 + 1)^4}{(x^2 + 1)^{10}}$

$\frac{2x(x^2 - 1)^3(x^2 + 1)^4[4(x^2 + 1) - 5(x^2 - 1)]}{(x^2 + 1)^{10}} \Rightarrow \frac{2x(x^2 - 1)^3(x^2 + 1)^4[4x^2 + 4 - 5x^2 + 5]}{(x^2 + 1)^{10}}$

$\frac{2x(x^2 - 1)^3(x^2 + 1)^4(-x^2 + 9)}{(x^2 + 1)^{10}} \Rightarrow \boxed{\frac{2x(x^2 - 1)^3(-x^2 + 9)}{(x^2 + 1)^6}}$

Final Answer. Whew!

Cancel the 4 (x^2+1)'s on the top with 4 of the 10 (x^2+1)'s on the bottom.



$f(x) = -x^2 + 6x + 7$

To find the slope of our red line:

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Where $f(x) = -x^2 + 6x + 7$, so

$f(x+h) = -(x+h)^2 + 6(x+h) + 7$

$$m = \lim_{h \to 0} \frac{-(x+h)^2 + 6(x+h) + 7 - (-x^2 + 6x + 7)}{h}$$

$$m = \lim_{h \to 0} \frac{-(x^2 + 2xh + h^2) + 6(x+h) + 7 - (-x^2 + 6x + 7)}{h}$$

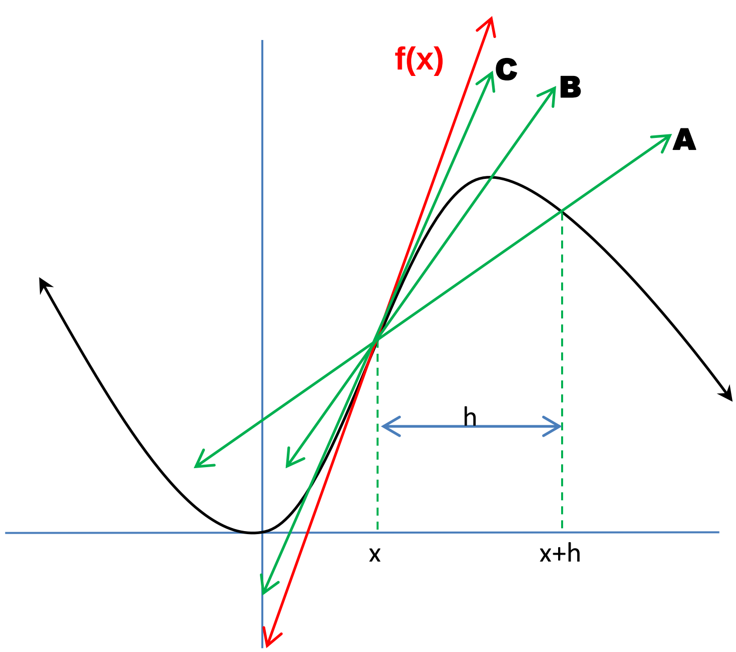
$$m = \lim_{h \to 0} \frac{-x^2 - 2xh - h^2 + 6x + 6h + 7 + x^2 - 6x - 7}{h}$$

$m = \lim_{h \to 0} \frac{-2xh - h^2 + 6h}{h}$ → factor out the common 'h' on top:

$m = \lim_{h \to 0} \frac{h(-2x - h + 6)}{h}$ → cancel the 'h' top and bottom:

Apply the limit (that is, let h=0): $m = (-2x - 0 + 6)$

So, our slope is: $\mathbf{m = -2x + 6}$, for any x we choose



Slope of **f(x)**:

$$m = \frac{f(x+h) - f(x)}{(x+h) - x}$$

-OR-

$$m = \frac{f(x+h) - f(x)}{h}$$

As h gets smaller (from **A** to **B** to **C**), the green line slope gets closer to matching the red line slope. When h = 0, the green line becomes the red line. The problem is that h ≠ 0, or we have a zero denominator, so we cheat and define the red line slope as:

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$