Supplement 12
Implicit Differentiation and the Chain Rule

An offshore oil well is leaking oil onto the ocean surface, forming a circular oil slick about .005 meters thick. If the radius of the slick is r meters, then the volume of oil spilled is $\mathbf{V = .005\pi r^2}$ cubic meters. Suppose that the oil is leaking at a constant rate of 20 cubic meters per hour. Find the rate at which the radius of the oil slick is increasing at a time when the radius is 50 meters.

$\frac{dV}{dt} = 2(.005)\pi r \frac{dr}{dt}$; $20 = .01\pi(50)\frac{dr}{dt}$; $\frac{20}{.01\pi(50)} = \frac{dr}{dt}$; $\frac{dr}{dt} = \boxed{\frac{40}{\pi}\text{ m/hr.}}$

2. Animal physiologists have determined experimentally that the weight, W (in kilograms), and the surface area, S (in square meters) of a typical horse are related by the empirical equation, $\boldsymbol{S = 0.1W^{\frac{2}{3}}}$. How fast is the surface area of a horse increasing at a time when the horse weighs 350 kg and is gaining weight at the rate of 200 kg per year?

$\frac{dS}{dt} = \frac{2}{3}(.1)W^{-\frac{1}{3}}\frac{dW}{dt}$; $\frac{dS}{dt} = \frac{2}{3}(.1)(350)^{-\frac{1}{3}}(200) = \boxed{1.89\text{ m}^2/\text{yr.}}$

3. Suppose that a tiny, experimental, kitchen appliance company's monthly sales and advertising expenses are approximately related by the equation, **xy – 6x + 20y = 0**, where x is dollars spent on advertising and y is dishwashers sold. Currently, the company is spending \$10 dollars on advertising and is selling 2 dishwashers each month. If the company plans to increase monthly advertising expenditures at the rate of \$1.50 per month, how fast will sales rise?

Sales will rise by 20%

$x\frac{dy}{dt} + y\frac{dx}{dt} - 6\frac{dx}{dt} + 20\frac{dy}{dt} = 0$

$10\left(\frac{dy}{dt}\right) + 2(1.5) - 6(1.5) + 20\left(\frac{dy}{dt}\right) = 0$; $30\left(\frac{dy}{dt}\right) - 4(1.5) = 0$; $\frac{dy}{dt} = \frac{6}{30} = \boxed{\frac{1}{5}}$

4. Suppose that the price, p (in dollars), and the weekly sales, x (in thousands of units), of a certain commodity satisfy the demand equation, $\mathbf{2p^3 + x^2 = 4500}$. Determine the rate at which sales are changing at a time when x = 50, p = 10, and the price is falling at the rate of \$.50 per week.

$6p^2\frac{dp}{dt} + 2x\frac{dx}{dt} = 0$; $6(10)^2(-.5) + 2(50)\frac{dx}{dt}$; $-300 + 100\frac{dx}{dt} = 0$

$\boxed{\frac{dx}{dt} = 3\text{ thousand units/week}}$

5. Suppose that the price, p (in dollars), and the demand, x (in thousands of units), of a commodity satisfy the demand equation, **6p + x + xp = 94**. How fast is the demand changing at a time when x = 4, p = 9, and the price is rising at the rate of \$2/week?

$6\frac{dp}{dt} + \frac{dx}{dt} + p\frac{dx}{dt} + x\frac{dp}{dt} = 0$

$6(2) + 1\cdot\frac{dx}{dt} + 9\cdot\frac{dx}{dt} + 4(2) = 0$; $10\frac{dx}{dt} + 20 = 0$; $\boxed{\frac{dx}{dt} = -2\text{ Thousand units per week}}$

6. Suppose that in Boston, the wholesale price, p, of oranges (in dollars per crate), and the daily supply, x crates, are related by the equation, **px + 7x + 8p = 328**. If there are 4 crates available today at a price of \$25 per crate, and if the supply is changing at the rate of $-\mathbf{0.3}$ crates per day, at what rate is the price changing?

$x\frac{dp}{dt} + p\frac{dx}{dt} + 7\frac{dx}{dt} + 8\frac{dp}{dt} = 0$

$4\frac{dp}{dt} + 25(-.3) + 7(-.3) + 8\frac{dp}{dt} = 0$; $12\frac{dp}{dt} - 9.6 = 0$; $\boxed{\frac{dp}{dt} = \$.80/\text{day}}$

7. Under certain conditions (called adiabatic expansion) the pressure, P, and the volume, V, of a gas such as oxygen satisfy the equation, $P^5V^7 = k$, where k is a constant. Suppose that at some moment, the volume of the gas is 4 liters, the pressure is 200 units, and the pressure is increasing at the rate of 5 units per second. Find the (time) rate at which the volume is changing.

$$\left(5P^4\frac{dV}{dt}\right)V^7 + P^5\left(7V^6\frac{dV}{dt}\right) = 0$$

$$5(200)^4(5)(4)^7 + (200)^5(7)(4)^6\frac{dV}{dt} = 0$$

$$25(200)^4(4)^7 = -(200)^5(7)(4)^6\frac{dV}{dt}$$

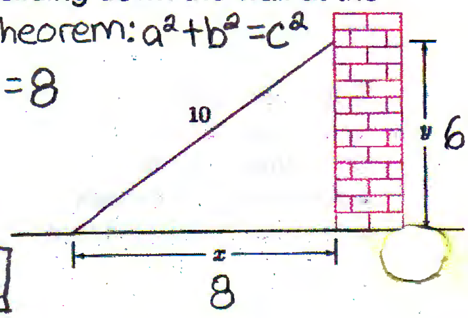
$$\frac{25(200)^4(4)^7}{-7(200)^5(4)^6} = \frac{25(4)}{-7(200)} = \boxed{-\frac{1}{14} = \frac{dV}{dt}}$$

8. The volume, V, of a spherical cancer tumor is given by, $V = \frac{\pi x^3}{6}$, where x is the diameter of the tumor. A physician estimates that the diameter is growing at the rate of 0.4 mm per day, at a time when the diameter is already 10 mm. How fast is the volume of the tumor changing at that time?

$$\frac{dV}{dt} = \frac{1}{6}\pi(3x^2)\frac{dx}{dt} = \frac{1}{6}\pi\cdot 3(10)^2(.4) = \frac{3\cdot 100\cdot 4\cdot\pi}{6\cdot 10} = \boxed{20\pi \text{ mm}^3/\text{day}}$$

$$.4 = \frac{4}{10}$$

9. Referring to the accompanying diagram, suppose that the foot of a 10' ladder is being pulled along the ground that the rate of 3 ft./sec. How fast is the top end of the ladder sliding down the wall at the time when the foot of the ladder is 8 feet from the wall? Pythagorean Theorem: $a^2+b^2=c^2$



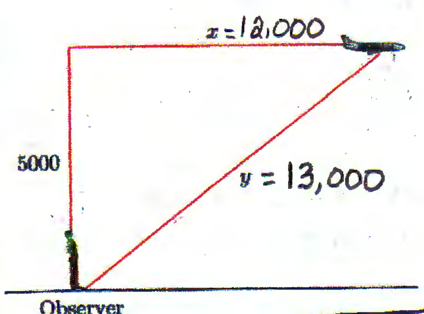
$$x^2 + y^2 = 10^2 \Rightarrow 8^2 + y^2 = 10^2;\ y^2 = 36;\ y = 6 \text{ when } x = 8$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(8)(3) + 2(6)\frac{dy}{dt} = 0;\ 12\frac{dy}{dt} = -48;\ \boxed{\frac{dy}{dt} = -4 \text{ ft./sec}}$$

Sliding DOWN

10. An airplane flying at 390 ft./sec. $\left(\frac{dx}{dt}\right)$ at an altitude of 5000 feet flew directly over an observer. Referring to the accompanying diagram, how fast is the distance from the observer to the airplane $\left(\frac{dy}{dt}\right)$ changing at the time when the airplane is 13,000 feet (y = 13,000) from the observer?



$$5000^2 + x^2 = y^2 \Rightarrow 5000^2 + x^2 = 13{,}000^2$$

$$x^2 = 144{,}000{,}000$$

$$x = 12{,}000$$

when $y = 13{,}000$

$$0 + 2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

$$2(12{,}000)(390) = 2(13{,}000)\frac{dy}{dt}$$

$$9{,}360{,}000 = 26{,}000\frac{dy}{dt}$$

$$\boxed{360 \text{ ft./sec} = \frac{dy}{dt}}$$

Answers

1. $\frac{40}{\pi}$ m/hr.
2. 1.89 m^2/yr.
3. $\frac{1}{5}$ dishwasher/month
4. \$3 thousand/week
5. − \$2 thousand/week
6. \$.80/crate
7. .07 liters/sec or $\frac{1}{14}$ liters/sec
8. 20 π mm/day
9. 4 ft./sec.
10. 360 ft./sec.