MOMENT OF INERTIA

In rotational motion, the moment of inertia is the equivalent to the mass in translational motion. *It measures the resistance of a body to an Angular Acceleration.*

The equivalent equation to $\mathbf{F} = m\mathbf{a}$ for a rotating body is $\mathbf{M} = I\boldsymbol{\alpha}$, where I is the moment of inertia.

$$I = \int_m r^2\,dm = \int_V r^2 \rho\, dV$$

Second moment about the z-axis

For constant ρ, we have

$$I = \rho \int_V r^2 dV$$

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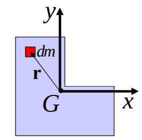
If the axis goes through the center of mass G and is perpendicular to the plane of motion, the moment of inertia is denoted by I_G



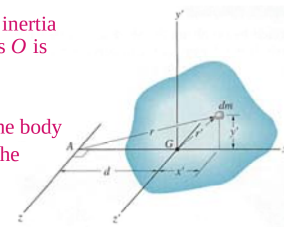
The units of I are $kg \cdot m^2$ or $slug \cdot ft^2$

THE PARALLEL AXES THEOREM

If I_G is known, the moment of inertia I_O about any other parallel axis O is given by

$$I_O = I_G + m\,d^2$$

Where m is the total mass of the body and d is the distance between the parallel axes.



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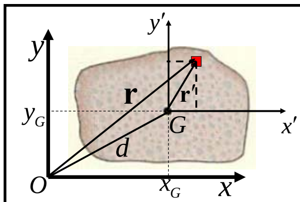

$$x_G = \frac{\int_m x\,dm}{\int_m dm} \qquad y_G = \frac{\int_m y\,dm}{\int_m dm}$$

$$I_O = \int_m r^2 dm = \int_m \left[(x_G + x')^2 + (y_G + y')^2 \right] dm$$

$$I_O = \int_m \left(x_G^2 + 2x_G x' + x'^2 + y_G^2 + 2y_G y' + y'^2 \right) dm$$

$$I_O = \left(x_G^2 + y_G^2 \right) \int_m dm + 2x_G \underbrace{\int_m x' dm}_{0} + 2y_G \underbrace{\int_m y' dm}_{0} + \underbrace{\int_m \left(x'^2 + y'^2 \right) dm}_{I_G}$$

$$I_O = d^2 m + I_G$$

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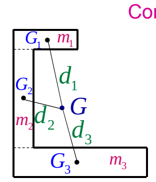
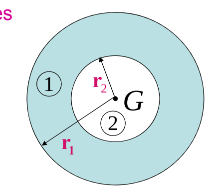
Radius of Gyration

Sometimes the moment of inertia about an axis is given as the *Radius of Gyration*, k in the form

$$I = m k^2 \quad \text{or} \quad k = \sqrt{\frac{I}{m}}$$

Composite bodies





$$I_G = \sum_{i=1}^{3} \left(I_{G_i} + m_i d_i^2 \right) \qquad I_G = I_{G_1} - I_{G_2}$$

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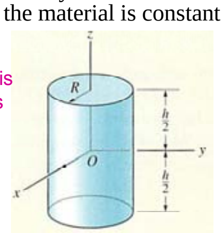
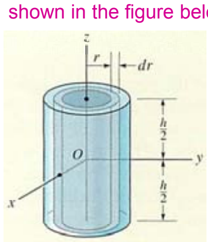
EXAMPLES

1. Determine the moment of inertia of the cylinder shown about the z-axis. The density ρ of the material is constant

SOLUTION

In this type of problem we create what is called a *shell element* of volume *dV* as shown in the figure below





The volume of the element is

$$dV = (2\pi r)(h)(dr)$$

and the mass of the element is

$$dm = \rho dV = \rho(2\pi r h)\,dr$$

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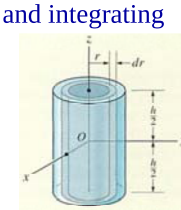
Hence $dI_z = r^2 dm = 2\pi\rho h r^3 dr$ and integrating

$$I_z = \int_m r^2 dm = 2\pi\rho h \int_0^R r^3 dr = \frac{\pi\rho h}{2} R^4$$



The mass of the cylinder is $m = \int_m dm = 2\pi\rho h \int_0^R r\,dr = \pi\rho h R^2$

So in terms of the mass $\mathrm{I}_z = \frac{1}{2} m R^2$

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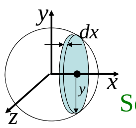
2. The sphere is formed by revolving the shaded area around the x-axis. Determine the moment of inertia I_x and express the result in terms of the total mass m of the sphere. The maerial has a constant density ρ.

SOLUTION

Define a differential disk with volume

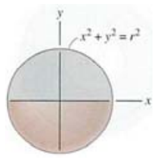


$dV = \pi y^2 dx$

Now $y^2 = r^2 - x^2$

So $dV = \pi(r^2 - x^2)dx$

and $dm = \rho\pi(r^2 - x^2)dx$



So the mass m is

$m = 2\rho\pi\int_0^r (r^2 - x^2)\,dx = 2\rho\pi(r^3 - \frac{r^3}{3})$ $\quad m = \rho\frac{4}{3}\pi r^3$

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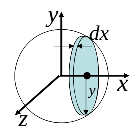
Using the fact that we can look at the ring as a thin cylinder:

$dI_x = \frac{1}{2}y^2 dm$



$dI_x = \frac{1}{2}\left(r^2 - x^2\right)\times\rho\pi\left(r^2 - x^2\right)dx$

$I_x = \frac{1}{2}\rho\pi\int_0^r \left(r^2 - x^2\right)^2 dx$

$I_x = \frac{8}{15}\rho\pi r^5$

$I_x = \frac{2}{5}m r^2$

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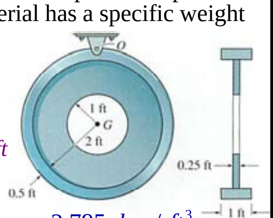
3. Determine the moment of inertia of the assembly about an axis which is perpendicular to the plane and passes through the point O. The material has a specific weight of $\gamma = 90\,lb/ft^3$

SOLUTION

1) OUTER RING:



$R_O = 2.5\,ft\ ,\ R_I = 2.0\,ft\ ,\ d_O = 1\,ft$

$I_G^{(1)} = \frac{1}{2}m_O R_O^2 - \frac{1}{2}m_I R_I^2$

$\rho = 2.795\,slug/ft^3$

$I_G^{(1)} = \frac{1}{2}\left(\rho\pi R_O^2 d_O\right)R_O^2 - \frac{1}{2}\left(\rho\pi R_I^2 d_I\right)R_I^2 = \frac{1}{2}\rho\pi d_O(R_O^4 - R_I^4)$

$I_G^{(1)} = \frac{1}{2}\times 2.795\times 3.14159\times 1\times(2.5^4 - 2^4)$

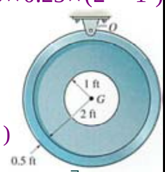
$I_G^{(1)} = 101.253\,slug/ft^2$

2) INNER RING: $r_o = 2\,ft\ ,\ r_i = 1\,ft\ ,\ d_I = 0.25\,ft$

$I_G^{(2)} = \frac{1}{2}m_o r_o^2 - \frac{1}{2}m_i r_i^2 = \frac{1}{2}\left(\rho\pi r_o^2 d_I\right)r_o^2 - \frac{1}{2}\left(\rho\pi r_i^2 d_I\right)r_i^2$

$I_G^{(2)} = \frac{1}{2}\rho\pi d_I\left(r_o^4 - r_i^4\right) = \frac{1}{2}\times 2.795\times 3.14159\times 0.25\times(2^4 - 1^4)$

$I_G^{(2)} = 16.464\,slug/ft^2$



$I_G = I_G^{(1)} + I_G^{(2)} = 117.717$

$m = m_O + m_I = \rho\pi d_O(R_O^2 - R_I^2) + \rho\pi d_I(r_o^2 - r_i^2)$

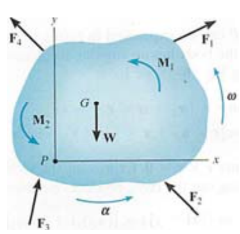
$m = 2.795\times 3.14159\times\left[1\times(2.5^2 - 2^2) + 0.25\times(2^2 - 1^2)\right]$

$I_O = I_G + md^2\ ,\ m = 26.34\,slug\ ,\ d = 2.5\,ft$

$I_O = 117.717 + 26.34\times 2.5^2 \Rightarrow I_O = 282\,slug/ft^2$

KINETIC EQUATIONS OF MOTION

The geometry and loading are assumed to be such that we can assume that the motion is planar. This requires that the geometry and loading be symmetrical with respect to a fixed reference plane, p. 409.



The reference frame x-y-z mus be *INERTIAL* with the origin at some arbitrary point P not necessarily in the body.
The axes *do not rotate*, but can translate with a constant velocity

Translational Motion:

In this case the previous analysis for systems of particles applies, and we have:

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The sum of all the external forces acting on the body is equal to the mass of the body times the acceleration of its center of mass.

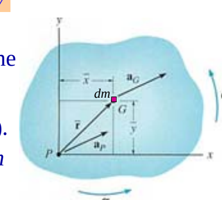
$$\sum \mathbf{F} = m\mathbf{a}_G$$

In the case of translational motion in the x-y plane we have

$$\sum F_x = m(a_G)_x$$
$$\sum F_y = m(a_G)_y$$

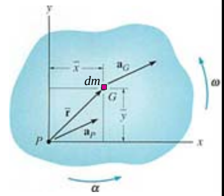
Rotational Motion:

Let the z-axis is at point P and $\mathbf{F}$ is the resultant $\left(\sum \mathbf{F}_i\right)$ of all forces acting on the body (ignoring internal forces). Then consider an element of mass dm at position $\mathbf{r}$ in the body:



$$d(\mathbf{r}\times\mathbf{F}) = \mathbf{r}\times dm\,\mathbf{a}$$
$$d\,\mathbf{M}_P = \mathbf{r}\times dm\,\mathbf{a}$$

If $\boldsymbol{\alpha}$ is the angular acceleration and $\boldsymbol{\omega}$ the angular velocity of the body, using $\mathbf{a}=\mathbf{a}_P+\boldsymbol{\alpha}\times\mathbf{r}-\omega^2\mathbf{r}$



$$d\mathbf{M}_P = dm\,\mathbf{r}\times\left(\mathbf{a}_P+\boldsymbol{\alpha}\times\mathbf{r}-\omega^2\mathbf{r}\right)$$
$$= dm\left[\mathbf{r}\times\mathbf{a}_P+\mathbf{r}\times(\boldsymbol{\alpha}\times\mathbf{r})-\omega^2(\mathbf{r}\times\mathbf{r})\right]$$
$$= dm\left[\mathbf{r}\times\mathbf{a}_P+\mathbf{r}\times(\boldsymbol{\alpha}\times\mathbf{r})\right]$$

In Cartesian coordinates:

$$dM_P\mathbf{k} = dm\left\{(x\mathbf{i}+y\mathbf{j})\times(a_{Px}\mathbf{i}+a_{Py}\mathbf{j})+(x\mathbf{i}+y\mathbf{j})\times[\alpha\mathbf{k}\times(x\mathbf{i}+y\mathbf{j})]\right\}$$
$$= dm(-ya_{Px}+xa_{Py}+\alpha x^2+\alpha y^2)\mathbf{k}$$
$$dM_P = dm(-ya_{Px}+xa_{Py}+\alpha r^2)$$

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And integrating over the whole body we get

$$M_P \equiv \sum M_P = a_{Py}\underbrace{\int_m x\,dm}_{m\bar{x}} - a_{Px}\underbrace{\int_m y\,dm}_{m\bar{y}} + \alpha\underbrace{\int_m r^2 dm}_{I_P}$$

Because $\bar{x}=\left(\int_m x\,dm\right)/m$ and $\bar{y}=\left(\int_m y\,dm\right)/m$

Thus $\sum M_P = -\bar{y}\,m a_{Px}+\bar{x}\,m a_{Py}+I_P\alpha$

If $P=G$, the equation reduces to

$$\sum M_G = I_G\alpha$$

The sum of the moments about the center of mass *G* of the body due to all external forces is equal to the product of the moment of inertia of the body about the center of mass G times the angular acceleration α of the body.

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Lets go back to $\sum M_P = -\bar{y}\,m a_{Px}+\bar{x}\,m a_{Py}+\alpha I_P$

Using the parallel axis theorem $I_P = I_G + m(\bar{x}^2+\bar{y}^2)$ we get

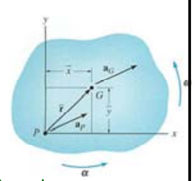
$$\sum M_P = \bar{y}\,m(-a_{Px}+\bar{y}^2\alpha)+\bar{x}\,m(a_{Py}+\bar{x}^2\alpha)+\alpha I_G \quad (*)$$

Now $\mathbf{a}_G = \mathbf{a}_P+\boldsymbol{\alpha}\times\bar{\mathbf{r}}-\omega^2\bar{\mathbf{r}}$ so

$$a_{Gx}\mathbf{i}+a_{Gy}\mathbf{j} = a_{Px}\mathbf{i}+a_{Py}\mathbf{j}+\alpha\mathbf{k}\times(\bar{x}\,\mathbf{i}+\bar{y}\,\mathbf{j})-\omega^2(\bar{x}\,\mathbf{i}+\bar{y}\,\mathbf{j})$$

and we get $a_{Gx} = a_{Px}-\bar{y}\alpha-\bar{x}\omega^2$

$a_{Gy} = a_{Py}+\bar{x}\alpha-\bar{y}\omega^2$



so that $-a_{Px}+\bar{y}\alpha = -a_{Gx}-\bar{x}\alpha$

$a_{Py}+\bar{x}\alpha = a_{Gy}+\bar{y}\omega^2$

Substituting these last expressions into (*) we have

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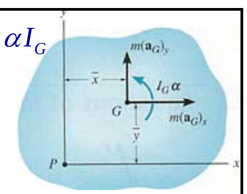
$$\sum M_P = -\bar{y}\,m a_{Gx}+\bar{x}\,m a_{Gy}+\alpha I_G$$

or

$$\sum M_P = \sum\left(\mathcal{M}_k\right)_P$$

Where $\left(\mathcal{M}_k\right)_P$ are called the *Kinetic Moments.*



The sum of the moments of the external forces about a point *P* are equivalent to the sum of the *Kinetic Moments* of the components of $m\mathbf{a}_G$ about *P* plus the *Kinetic Moment* of $I_G\boldsymbol{\alpha}$.

The components $ma_{Gx}\mathbf{i}$ and ma_{Gy} are treated as *Sliding Vectors* acting anywhere along their *line of action.*
$\alpha I_G\,\mathbf{k}$ is treated as a *Free Vector* and can act at *any point*.

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We must remember that $m\mathbf{a}_G$ and $I_G\boldsymbol{\alpha}$ are not *force or couple* moments. These are the *Effects* of forces and couples acting on the body

In scalar form,there are three independent equations of planar motion:

$$\sum F_x = m a_{Gx}$$
$$\sum F_y = m a_{Gy}$$
$$\sum M_G = \alpha I_G \quad \text{or} \quad \sum M_P = \sum\left(\mathcal{M}_k\right)_P$$

This underscores the need to always draw the free body diagram to account for $\sum F_x, \sum F_y, \sum M_G$ or $\sum M_P$.

We should also always draw the kinetic diagram to accont for the terms $m a_{Gx}, m a_{Gy}$ and αI_G.

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Equations of Motion in Pure Translation

Under translation only all points in a body have the same acceleration $\mathbf{a}=\mathbf{a}_G$ and the angular acceleration $\boldsymbol{\alpha}=\mathbf{0}$.

Rectilinear Translation:
In this case all points in the body travel along parallel staright line paths. The equations of motion become:

$$\sum F_x = m a_{Gx}$$
$$\sum F_y = m a_{Gy}$$
$$\sum M_G = 0$$

If the sum of the moments is taken about another point different from *G*, then the moment $m\mathbf{a}_G$ must be taken into account

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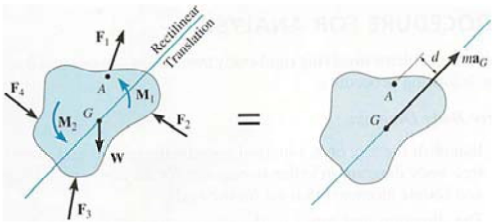
For example, if a point A as shown in the figure is chosen



$$\sum M_A = \sum \left(\mathcal{M}_k\right)_A = (m a_G) \cdot d$$

EXAMPLES

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