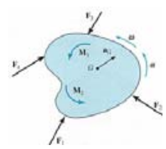
GENERAL PLANE MOTION

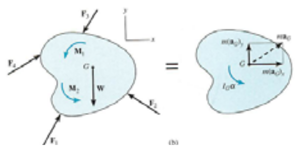
If an inertial x – y coordinate system is chosen, the three equations of motion may be written as:

$$\sum F_x = m\,a_{Gx}$$
$$\sum F_y = m\,a_{Gy}$$
$$\sum M_G = I_G \alpha$$



In some problems it is more convenient to sum moments about a point *P* different from *G* (e.g. to eliminate unknown forces). In this case the three equations of motion become:

$$\sum F_x = m\,a_{Gx}$$
$$\sum F_y = m\,a_{Gy}$$
$$\sum M_P = \sum (\mathcal{M}_k)_P$$

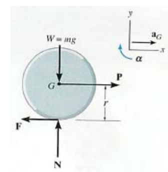


Frictional Rolling Problems

This is a special class of problems involving wheels, cylinders or similar bodies that roll on a **rough** plane surface.

Furthermore, it may not be known if the body rolls *without slipping* or if it *slides as it rolls.*

Consider a homogeneous disk of mass m and subjected to a known horizontal force P.



From the equations of motion we get:

$$\sum F_x = m\,a_{Gx} \;\rightarrow\; P - F = m\,a_{Gx}$$
$$\sum F_y = m\,a_{Gy} \;\rightarrow\; N - W = 0$$
$$\sum M_G = I_G \alpha \;\rightarrow\; F \cdot r = I_G\, \alpha$$

2

Here we have four unknowns, F, N, a_{Gx}, and α. Therefore we need one more equation.

Case 1: No Slip

No slip occurs if $F \le \mu_s N$

and in this case: $a_G = \alpha\, r$

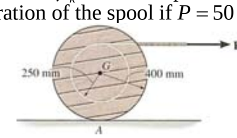
Case2: Slip

If $F > \mu_s N$ we must "re-work" the problem because there is slip. In this case α and a_G are not independent of each other. The extra relation is $F = \mu_k N$

EXAMPLES

3

1. The spool has a mass of 100 *kg* and a radius of gyration $k_G = 0.3m$. If the coefficients of static and dynamic friction at *A* are $\mu_s = 0.2$ and $\mu_k = 0.15$, respectively determine the angular acceleration of the spool if $P = 50\ N$ and is *directed upwards.*



Data:
$m = 100\,kg$
$W = 981\,N$
$k_G = 0.3m$
$\mu_s = 0.2$
$\mu_k = 0.15$

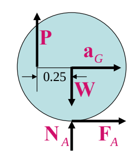
SOLUTION

1) First find the moment of inertia

$$I_G = m k_G^2 = 100 \times 0.3^2 = 9\,kg \cdot m^2$$

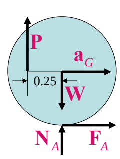
2) Equilibrium equations

$$\sum F_x = m\,a_{Gx} \;\Rightarrow\; F_A = 100\,a_{Gx}$$
$$\sum F_y = 0 \;\Rightarrow\; 50 + N_A - 981 = 0$$



4

$$\sum M_G = I_G \alpha \;\Rightarrow\; -50 \times 0.25 + F_A \times 0.4 = 9\alpha$$



So independently of whether we have slip or no-slip:

$F_A = 100\,a_{Gx}$ (1)
$N_A = 931$ (2)
$0.4 F_A - 9\alpha = 12.5$ (3)

3) Assuming No-Slip: $a_{Gx} = 0.4\alpha$ (4)

And solving the equations:

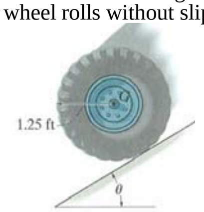
$\alpha = 0.5\,rad/s^2$
$a_{Gx} = 0.2\,m/s^2$
$N_A = 931\,N$
$F_A = 20\,N$

4) Check if the no-slip assumption is correct:

$$(F_A)_{\max} = \mu_s N_A = 0.2 \times 931 = 186.2 > F_A \;\text{ OK}$$

5

2. The wheel has a weight of 30 lb and a radius of gyration of $k_G = 0.6\ ft$. If the coefficients of static and dynamic friction between the wheel and the plane are μ_s=0.2 and $\mu_k = 0.15$, determine the maximum angle θ of the inclined plane so that the wheel rolls without slipping



Data:
$W = 30\,lb$
$m = 0.93168\,slug$
$\mu_s = 0.2$
$\mu_k = 0.15$
$\theta_{\max} = ?$

SOLUTION

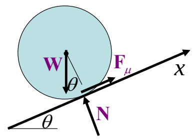
1) Moment of Inertia:

$$I_G = m k_G^2 = 0.93168 \times 0.6^2 = 0.3354\,slug \cdot ft^2$$

6

2) Equilibrium Equations:

$$\sum F_x = m a_{Gx} \Rightarrow F_\mu - W\sin\theta = m a_{Gx}$$
$$\sum F_y = 0 \Rightarrow -W\cos\theta + N = 0$$
$$\sum M_G = I_G\alpha \Rightarrow F_\mu \times r = I_G\alpha$$



3) For no-slip $a_{Gx} = r\alpha$ and max θ occurs when $F_\mu = \mu_s N$

The equilibrium equations become:

$$0.2N - 30\sin\theta = 0.93168\times(1.25\alpha) \quad (1)$$
$$N = 30\cos\theta \quad (2)$$
$$0.2N\times1.25 = 0.3354\alpha \quad (3)$$

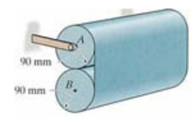
From (3) $\alpha = \dfrac{0.2\times1.25}{0.3354}N = 0.74538\times30\cos\theta = 22.361\cos\theta$

From (1) $6\cos\theta - 30\sin\theta = -26.042\cos\theta \Rightarrow \tan\theta = 1.0681$

$$\theta_{\max} = 46.9^\circ \quad (N = 20.5\,lb, a_{Gx} = 19.1\,ft/s^2, \alpha = 15.3\,rad/s^2)$$

3. Along strip of paper is wrapped into two rolls, each having a mass of 8 *kg*. Roll *A* is pin supported about its center whereas roll *B* is not centrally supported. If *B* is brought into contact with *A* and released from rest, determine the initial tension in the paper between the rolls and the angular acceleration of each roll. For the calculation, assume the rolls to be approximated by cylinders.

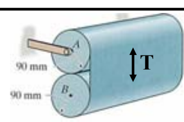
Data:
$m_A = m_B = 8\,kg$
$W_A = W_B = 78.48\,N$
$r_A = r_B = 0.09\,m$



SOLUTION

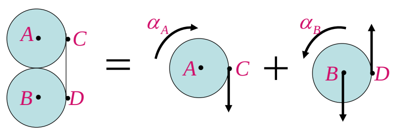
1) Moments of Inertia:

$I_A = I_B = I = \frac{1}{2}mr^2 = \frac{1}{2}\times8\times0.09^2$
$I = 0.0324\,kg\cdot m^2$



$\alpha_A = ?$
$\alpha_B = ?$
$a_B = ?$
$T = ?$

Initially we must have $\omega_A = \omega_B = 0$



2) Roll A: $\sum M_A = I_A\alpha_A \Rightarrow -T\times0.09 = -0.0324\alpha_A$

$$T = 0.36\alpha_A \quad (1)$$

3) Roll B: $\sum F_y = m a_{By} \Rightarrow T - W = m a_{By}$

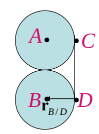
$$T - 78.48 = -8a_{By} \quad (2)$$

$\sum M_B = I_B\alpha_B \Rightarrow T\,r = I_B\alpha_B \Rightarrow 1.5T = 0.0324\alpha_B$

$$T = 0.36\alpha_B \quad (3)$$

4) We need one more equation, look at the kinematics:

$\mathbf{a}_D = \mathbf{a}_C = -\alpha_A r\,\mathbf{j} = -0.09\alpha_A\mathbf{j}$
$\mathbf{a}_B = \mathbf{a}_D + \boldsymbol{\alpha}_B \times \mathbf{r}_{B/D}$
$-a_B\,\mathbf{j} = -0.09\alpha_A\,\mathbf{j} + (\alpha_B\mathbf{k})\times(-0.09\mathbf{i})$



$$a_B = 0.09\times(\alpha_A + \alpha_B) \quad (4)$$

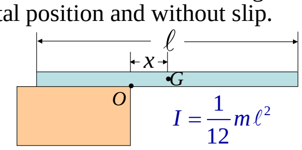
From (1) and (3) $\alpha_B = -\alpha_A = \alpha$

Substituting in (4) $a_B = 0.18\alpha$

Replacing into (2) $0.36\alpha - 78.48 = -8\times(0.18\alpha)$

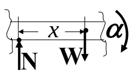
$$\alpha = 4.36\,rad/s^2 \qquad a_B = 7.85\,m/s^2$$
$$T = 1.57\,N$$

4. A slender bar of mass *m* and length ℓ rests on an ledge as shown. Find the length *x* that maximizes the angular acceleration α when the bar is let go from rest in the horizontal position and without slip.



$$I = \frac{1}{12}m\ell^2$$

SOLUTION

FBD 

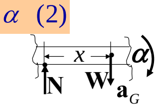
$$a_{Gy} = x\alpha$$

1) Equilibrium in the y-direction:

$$\sum F_y = N - mg = -ma_{Gy} = N = mg - mx\alpha \quad (1)$$

2) Moments about G:

$$\sum M_G = -I_G\,\alpha \Rightarrow -N\,x = -I_G\,\alpha \quad (2)$$



3) Substitute (1) into (2) and maximize.

$$(mg - mx\alpha)x = I_G\alpha \rightarrow (mx^2 + I_G)\alpha = mgx$$

Differentiate α with respect to x:

$$2mx\alpha + (mx^2 + I_G)\frac{d\alpha}{dx} = mg$$

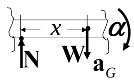
$$\frac{d\alpha}{dx} = \frac{mg - 2mx\alpha}{mx^2 + I_G} = 0 \Rightarrow x = \frac{g}{2\alpha} \quad (3)$$

4) Solve the equations

From (1): $N = mg - m\dfrac{g}{2\alpha}\alpha \Rightarrow N = \dfrac{mg}{2}$

From (2): $\frac{mg}{2}\cdot\frac{g}{2\alpha}=\frac{1}{12}m\ell^2\alpha \Rightarrow \alpha^2=\frac{12g^2}{4\ell^2} \Rightarrow \alpha=\frac{\sqrt{3}g}{\ell}$

Finally $x=\frac{\ell}{2\sqrt{3}}$



If instead of taking moments about G we do it about O:

$$\sum M_O=-mgx=-mxa_G-I_G\alpha \Rightarrow -mgx=-mx^2\alpha-I_G\alpha$$

Hence $(mx^2+I_G)\alpha=mgx$, the same as above.

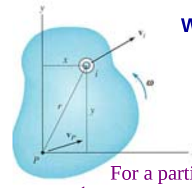
Check that x gives a max. Using $\alpha=Nx/I_G$ we get

$$\frac{d^2\alpha}{dx^2}=\frac{-4mNx(mx^2+I_G)-2mx(mgI_G-2mNx^2)}{I_G(mx^2+I_G)^2}$$

$$\frac{d^2\alpha}{dx^2}=\frac{-4mNxI_G-2m^2g\,xI_G}{I_G(mx^2+I_G)^2}<0$$

13

WORK AND ENERGY



Consider a planar body as shown. First we need to find the kinetic energy of the body including *ROTATIONAL* kinetic energy

For a particle i of mass dm in the body, the kinetic energy is $T=\frac{1}{2}v_i^2\,dm$ and the kinetic energy for the entire body is

$$T=\frac{1}{2}\int_m v_i^2\,dm.$$

Expressing v_i in terms of the velocity of point P we have:

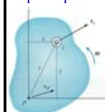
$$\mathbf{v}_i=\mathbf{v}_P+\mathbf{v}_{i/P}=v_{Px}\mathbf{i}+v_{Py}\mathbf{j}+(\omega\mathbf{k})\times(x\,\mathbf{i}+y\,\mathbf{j})$$

or $\mathbf{v}_i=(v_{Px}-\omega y)\mathbf{i}+(v_{Py}+\omega x)\mathbf{j}$

and taking the square of the magnitude

14

$$\mathbf{v}_i\cdot\mathbf{v}_i=v_i^2=(v_{Px}-\omega y)^2+(v_{Py}+\omega x)^2$$



$$=v_{Px}^2-2v_{Px}\omega y+\omega^2y^2+v_{Py}^2+2v_{Py}\omega x+\omega^2x^2$$

$$=v_P^2-2v_{Px}\omega y+2v_{Py}\omega x+\omega^2r^2$$

Substitute into the expression for the total kinetic energy to get:

$$T=\frac{1}{2}\underbrace{\left(\int_m dm\right)}_{m}v_P^2-v_{Px}\omega\underbrace{\left(\int_m y\,dm\right)}_{\bar{y}m}+v_{Py}\omega\underbrace{\left(\int_m x\,dm\right)}_{\bar{x}m}+\frac{1}{2}\omega^2\underbrace{\left(\int_m r^2dm\right)}_{I_P}$$

Therefore $T=\frac{1}{2}mv_P^2-v_{Px}\omega\bar{y}m+v_{Py}\omega\bar{x}m+\frac{1}{2}I_P\omega^2$

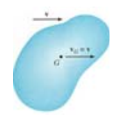
and if $P=G$ $\quad T=\frac{1}{2}mv_G^2+\frac{1}{2}I_G\omega^2$

15

Kinetic energy of a planar rigid body:

1) Translational kinetic energy:

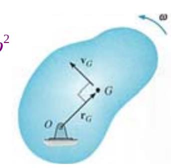
$$T=\frac{1}{2}mv_G^2$$



2) Rotation about a fixed axis

$$T=\frac{1}{2}mv_G^2+\frac{1}{2}I_G\omega^2$$

Or from $v_G=r_G\omega$ we have $T=\frac{1}{2}(I_G+mr_G^2)\omega^2$ so using the parallel axis theorem

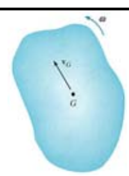


$$T=\frac{1}{2}m\,I_O\omega^2$$

16

3) Kinetic Energy in General Plane Motion

$$T=\frac{1}{2}mv_G^2+\frac{1}{2}I_G\omega^2$$



Note: Because the energy is a scalar, the total kinetic energy for a system of connected rigid bodies is the SUM of the kinetic energies of each of the moving parts

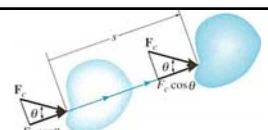
REVIEW: Work of a Force.

Work of a variable force **F**:

$$U_F=\int_s \mathbf{F}\cdot d\mathbf{r}=\int_s F\cos\theta\,ds$$

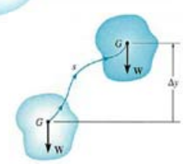
Work of a constant force:

$$U_{F_C}=(F_C\cos\theta)s$$



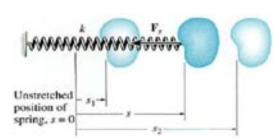
Work of a Weight:

$$U_W=-W\Delta y$$



Work of a Spring Force:

$$U_s=-\left(\frac{1}{2}ks_2^2-\frac{1}{2}ks_1^2\right)$$



The Work of a Couple

A couple of forces acting on a body produce a moment, and a moment is always equivalent to a couple, so we can write

$$M = F\,r$$

Because the forces are equal and in opposite direction, the work done by the forces when the body undergoes a translation cancels out.

Translation

If the body undergoes a rotation $d\theta$, each force undergoes a displacement $ds = (r/2)d\theta$.
Hence the total work done by the forces is

$$dU_M = F\left(\frac{r}{2}d\theta\right) + F\left(\frac{r}{2}d\theta\right) = F\,r\,d\theta = M\,d\theta$$

Rotation

19

The work done by a moment M when a plane body rotates an angle from θ_1 to θ_2 *measured in radians* is

$$U_M = \int_{\theta_1}^{\theta_2} M\,d\theta$$

If the moment **M** is constant, then $U_M = M(\theta_2 - \theta_1)$

The Principle of Work and Energy

$$T_1 + \sum U_{1-2} = T_2$$

Remains the same, but now the kinetic energy also contains rotational energy, and the work includes the work done by moments.

If we have several rigid bodies connected by pins, inextensible cables or in mesh with each other, the principle applies to the entire system, and the work done by internal forces cancels out.

20