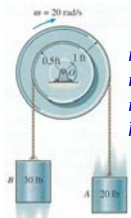
EXAMPLES

1. The double pulley consists of two parts that are attached to one another. It has a weight of 50 *lb* and a centroidal radius of gyration of $k_O = 0.6\ ft$ and is turning with an angular velocity of 20 rad/s clockwise. Determine the kinetic energy of the system. Assume that neither cable slips on the pulley.

SOLUTION



Data:

$m_O = 1.5528\,slug$

$m_A = 0.62112\,slug$

$m_B = 0.93168\,slug$

$k_O = 0.6$

$$T = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 + \frac{1}{2}I_O\omega_O^2$$

$$v_A = r_A\omega_O = 1\times 20 = 20\ ft/s$$

$$v_B = r_B\omega_O = 0.5\times 20 = 10\ ft/s$$

$$\frac{1}{2}m_A v_A^2 = \frac{1}{2}\times 0.62112\times 20^2 = 124\ lb\cdot ft$$

$$\frac{1}{2}m_B v_B^2 = \frac{1}{2}\times 0.93168\times 10 = 46.6\,lb\cdot ft$$

$$\frac{1}{2}I_O\omega_O^2 = \frac{1}{2}\times 1.5528\times 0.6^2\times 20^2 = 111.8\,lb\cdot ft$$

$$T = 283\,lb\cdot ft$$

2. The soap-box car has a weght of 110 *lb*, including the passanger but *excluding* its four wheels. Each wheel has a weight of 5 *lb*, radius of 0.5 *ft*, and a radius of gyration $k = 0.3\ ft$, computed about an axis passing through the wheel's axle. Determine the car's speed after it has traveled 100 *ft* starting from rest. The wheels roll without slipping. Neglect air resistance.

Data:

$W_C = 110\,lb$

$W_w = 5\,lb$

$m_C = 3.4161\,slug$

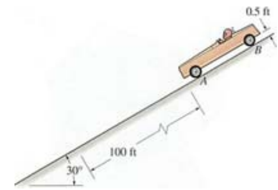
$m_w = 0.15528\,slug$

$r_w = 0.5\ ft$

$k = 0.3\ ft$

$s = 100\ ft$

$v_s = ?$



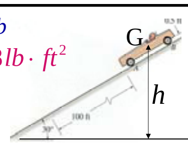
SOLUTION

2

The total weight is $W_T = 110 + 4\times 5 = 130\,lb$

$$I_w = m_w k^2 = 0.15528\times 0.3^2 = 0.01398\,lb\cdot ft^2$$

$$h = 100\sin 30^\circ = 50\ ft$$



$$T_1 + \sum U_{1-2} = T_2$$

$$T_1 = 0$$

$$\sum U_{1-2} = W_T h = 6500\,lb\cdot ft$$

$$T_2 = 4\times\left(\underbrace{\frac{1}{2}I_w\omega^2}_{\text{wheel rotation}} + \underbrace{\frac{1}{2}m_w v^2}_{\text{wheel translation}}\right) + \underbrace{\frac{1}{2}m_C v^2}_{\text{body translation}}$$

$$T_2 = 4\times\frac{1}{2}\times\left(0.01398\left(\frac{v}{0.5}\right)^2 + 0.15528v^2\right) + \frac{1}{2}\times 3.4161v^2$$

3

$$T_2 = \underbrace{0.31056v^2}_{\text{wheels rotation}} + \underbrace{0.1118v^2}_{\text{wheels translation}} + 1.7081v^2$$

$$0 + 6500 = 2.1305v^2$$

$$v = 55.2\ ft/s$$

NOTE:

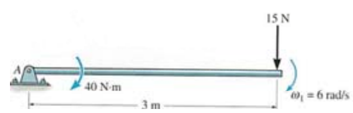
If we do not accout for rotational energy then $v = 59.8\ ft/s$ (8% difference)

3. The 4 *kg* slender rod is subjected to the force and couple moment. When it is in the position shown it has an angular velocity $\omega_1 = 6\,rad/s$. Determine its angular velocity at the instant it has rotated downwards 90°. The force is always applied perpendicular to the axis of the rod. Motion occurs in the vertical plane.

Data: $m = 4\,kg$

$W = 39.24\,N$

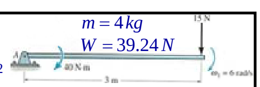
$\omega_2 = ?$



SOLUTION

$m = 4\,kg$

$W = 39.24\,N$



$$I_A = \frac{1}{3}m\ell^2 = \frac{1}{3}\times 4\times 3^2 = 12\,kg\cdot m^2$$

$$T_1 + \sum U_{1-2} = T_2$$

If we consider moments about A then only the rotational energy is needed.

$$T_1 = \frac{1}{2}I_A\omega_1^2 = \frac{1}{2}\times 12\times 6^2 = 216\,N\cdot m$$

$$\sum U_{1-2} = M(\theta_2 - \theta_1) + F\,r(\theta_2 - \theta_1) + W(r/2) = (-40)\times(-\pi/2) + (-15)\times 3\times(\pi/2) + 39.24\times 1.5 = 192.39\,N\cdot m$$

$$T_2 = \frac{1}{2}I_A\omega_2^2 = \frac{1}{2}\times 12\times\omega_2^2 = 6\omega_2^2$$

$$408.39 = 6\omega_2^2 \quad\Rightarrow\quad \omega_2 = 8.25\,rad/s$$

5

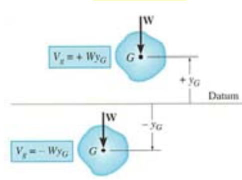
CONSERVATION OF ENERGY

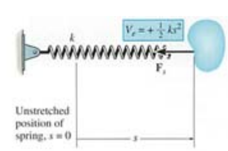
If all the forces acting on the system are conservative, then:

$$T_1 + V_1 = T_2 + V_2$$

Typically $V = V_g + V_e$ where

$$V_g = W\,y_G \qquad V_e = \frac{1}{2}k\,s^2$$

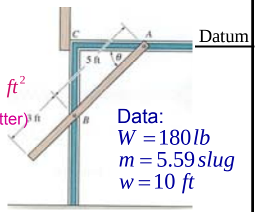




EXAMPLES

6

1. The door is made from one piece whose ends move along the horizontal and vertical tracks. If the door is in the open position $\theta = 0^\circ$, and then eleased, determine the speed at which its end A strikes the stop at C. Assume the door is a 180 lb thin plate having a width of 10 ft.



SOLUTION

$$I_G = \frac{1}{12} m \ell^2 = \frac{1}{12} \times 5.59 \times 8^2 = 29.813\, lb \cdot ft^2$$

(From tables, note that the width does not matter)

$y_G = -4\ ft$

(total drop of the mass center G)

Data:
$W = 180\, lb$
$m = 5.59\, slug$
$w = 10\ ft$

$$T_1 + V_1 = T_2 + V_2$$

$T_1 = 0$

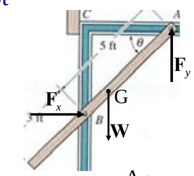
$V_1 = 0$

7

Note that the forces F_x and F_y do not perform any work.

Also, $|\mathbf{x}_G - \mathbf{x}_B| = 1\ ft$

FBD

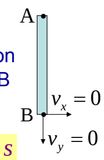


$$T_2 = \frac{1}{2} I_G \omega^2 + \frac{1}{2} m v_G^2 = \frac{1}{2} \times 29.81 \times \omega^2 + \frac{1}{2} 5.59 \times (1 \times \omega^2)$$

$T_2 = 17.7\, \omega^2$

$V_2 = -Wh = -180 \times 4 = -720\, lb \cdot ft$

Rotation about B



$$T_2 + V_2 = 0 = 17.7\,\omega^2 - 720 \Rightarrow \omega = 6.3779\, rad/s$$

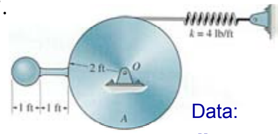
$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B} = (6.3779\mathbf{k}) \times (5\,\mathbf{j})$$

$$\mathbf{v}_A = -31.9\mathbf{i}$$

8

2. The disk A is pinned at O and weights 15 lb. A 1 ft rod weighting 2 lb and a 1 ft diameter sphere weighting 10 lb are welded to the disk, as shown. If ht spring is originally streched 1 ft and the sphere is released from the position shown, determine the angular velocity of the disk when it has rotated 90°.

SOLUTION



$$I_{G_D} = \frac{1}{2} m_D r_D^2 = \frac{1}{2} \times 0.46584 \times 2^2$$

$I_{G_D} = 0.9318\, slug \cdot ft^2$

$$I_{G_R} = \frac{1}{12} m_R \ell^2 = \frac{1}{12} \times 0.06211 \times 1^2$$

$I_{G_R} = 0.00518\, slug \cdot ft^2$

$$I_{G_S} = \frac{2}{5} m_S r_S^2 = \frac{2}{5} \times 0.31056 \times 0.5^2$$

$I_{G_S} = 0.03106\, slug \cdot ft^2$

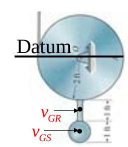
Data:
$W_D = 15\, lb$
$W_R = 2\, lb$
$W_S = 10\, lb$
$m_D = 0.46584\, slug$
$m_R = 0.06211\, slug$
$m_S = 0.31056\, slug$
$\theta = 90^\circ$
$\omega = ?$

9

$$T_1 + V_1 = T_2 + V_2$$

$T_1 = 0$

$$V_1 = \underbrace{\frac{1}{2} k \Delta s^2}_{\text{spring prestretch}} = \frac{1}{2} \times 4 \times 1^2 = 2\, lb \cdot ft$$



$$T_2 = T_D + T_R + T_S$$

$$T_D = \underbrace{\frac{1}{2} I_{G_D} \omega^2}_{\text{disk rotational energy}} = 0.4659\,\omega^2$$

$h_{GR} = 2.5\ ft$ $\quad v_{GR} = r_{GR}\omega = 2.5\omega$

$h_{GS} = 3.5\ ft$ $\quad v_{GS} = r_{GS}\omega = 3.5\omega$

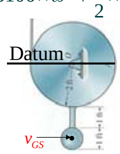
$$T_R = \underbrace{\frac{1}{2} I_{G_R} \omega^2}_{\text{rod rotational energy}} + \underbrace{\frac{1}{2} m_R v_{GR}^2}_{\text{rod translational energy}} = \frac{1}{2} \times 0.00518 \times \omega^2 + \frac{1}{2} \times 0.06211 \times (2.5\,\omega)^2$$

$T_R = 0.19668\,\omega^2$

10

$$T_S = \underbrace{\frac{1}{2} I_{G_S} \omega^2}_{\text{sphere rotational energy}} + \underbrace{\frac{1}{2} m_S v_{GS}^2}_{\text{sphere translational energy}} = \frac{1}{2} \times 0.03106 \times \omega^2 + \frac{1}{2} \times 0.31056 \times (3.5\,\omega)^2$$

$T_S = 1.9177\,\omega^2$



$h_{GS} = 3.5\ ft$

$v_{GS} = r_{GS}\omega = 3.5\omega$

Finally:

$T_2 = 0.4659\omega^2 + 0.19668\omega^2 + 1.9177\omega^2$

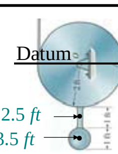
$T_2 = 2.5803\omega^2$

Potential energy at 2: $V_2 = V_{spring} + V_{rod} + V_{sphere}$

$$V_{spring} = \frac{1}{2} k (r\theta)^2 = \frac{1}{2} \times 4 \times \left(1 + 2 \times \frac{\pi}{2}\right)^2 = 34.306\, lb \cdot ft$$

11

$$V_{rod} = -W_R h_{GR} = -2 \times 2.5 = -5\, lb \cdot ft$$



$h_{GR} = 2.5\ ft$

$h_{GS} = 3.5\ ft$

$$V_{sphere} = -W_S h_{GS} = -10 \times 3.5 = -35\, lb \cdot ft$$

$$V_2 = 34.306 - 5 - 35 = -5.694\, lb \cdot ft$$

Put all together:

$$2 = 2.5803\omega^2 - 5.694$$

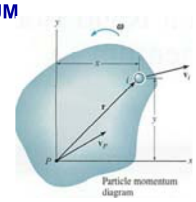
$$\omega = 1.73\, rad/s$$

12

IMPUSLE AND MOMENTUM

Linear momentum: $\mathbf{L} = m\,\mathbf{v}_G$

Angular Momentum: Consider a particle i of mass dm in a body rotating with angular velocity ω and where the point P has velocity $\mathbf{v}_P$

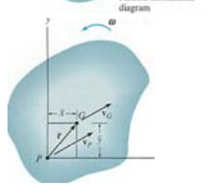


The vlocity of the particle i is

$$\mathbf{v}_i = \mathbf{v}_P + \boldsymbol{\omega} \times \mathbf{r}_{i/P}$$

The "moment" of the particle about P is

$$d(\mathbf{H}_P) = \mathbf{r}_{i/P} \times \mathbf{v}_i$$



In terms of the coordinate components:

$$d(H_P)\mathbf{k} = (x\,\mathbf{i} + y\mathbf{j}) \times dm[v_{Px}\mathbf{i} + v_{Py}\mathbf{j} + (\omega\mathbf{k}) \times (x\,\mathbf{i} + y\,\mathbf{j})]$$

13

Since all terms are in the **k** direction the scalar equation is

$$d(H_P) = -d\,m\,y\,v_{Px} + d\,m\,x\,v_{Py} + d\,m\,\omega r_i^2$$

Integrating over the whole body:

$$H_P = -\left(\underbrace{\int_m y\,dm}_{m\bar{y}}\right)v_{Px} + \left(\underbrace{\int_m x\,dm}_{m\bar{x}}\right)v_{Py} + \left(\underbrace{\int_m r_i^2\,dm}_{I_P}\right)\omega$$

So we get:

$$H_P = -m\,\bar{y}\,v_{Px} + m\,\bar{x}\,v_{Py} + I_P\omega$$

And if $P = G$

$$\mathbf{H}_G = I_G\,\boldsymbol{\omega}$$

14

The angular momentum of a body about its center of gravity G is the product of its moment of inertia I and its angular velocity **ω.**

To find an expression for the angular momentum about an arbitrary point P in terms of G, by the parallel axis theorem:

$$I_P = I_G + m(\bar{x}^2 + \bar{y}^{2)}$$

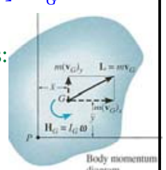
Substituting in the expression for H_P derived above:

$$(H_P = -m\,\bar{y}\,v_{Px} + m\,\bar{x}\,v_{Py} + I_P\omega)$$

$$H_P = \bar{y}\,m[-v_{Px} + \bar{y}\,\omega] + \bar{x}\,m[v_{Py} + \bar{x}\,\omega] + I_G\omega$$

Expressing $\mathbf{v}_G$ in terms of $\mathbf{v}_P$,
$\mathbf{v}_G = \mathbf{v}_P + \omega \times \mathbf{r}_{G/P}$ or in terms of components:

$$v_{Gx}\mathbf{i} + v_{Gy}\mathbf{j} = v_{Px}\mathbf{i} + v_{Py}\mathbf{j} + (\omega\mathbf{k}) \times (\bar{x}\,\mathbf{i} + \bar{y}\,\mathbf{j})$$



This yields:

$$v_{Gx} = v_{Px} - \bar{y}\,\omega$$
$$v_{Gy} = v_{Py} + \bar{x}\,\omega$$

The equation for H_P can then be written as:

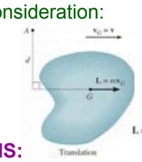
$$H_P = -\bar{y}\,m v_{Gx} + \bar{x}\,m v_{Gy} + I_G\,\omega$$

If the angular momentum is computed about a point P, it is equivalent to the linear momentum $m\mathbf{v}_G$ about P plus the angular momentum $I_G\boldsymbol{\omega}$.

Let us now examine linear and angular momentum for the three different types of motion under consideration:

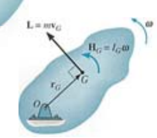
TRANSLATION:

$$L = m v_G$$
$$H_G = 0$$



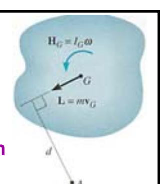
ROTATION ABOUT A FIXED AXIS:

$$L = m v_G$$
$$H_G = I_G\,\omega \quad \text{or} \quad H_O = I_O\,\omega$$



GENERAL PLANE MOTION:

$$L = m v_G$$
$$H_G = I_G\,\omega \quad \text{or} \quad H_A = I_G\,\omega + d\,m v_G$$



Principle of Linear Impulse and Momentum

$$m(v_G)_1 + \sum\int_{t_1}^{t_2}\mathbf{F}\,dt = m(v_G)_2$$

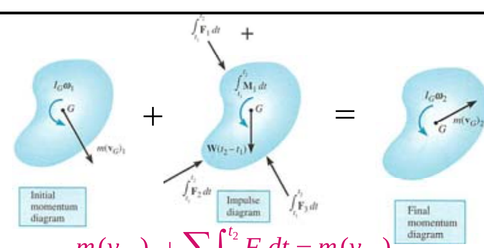
Principle of Angular Impulse and Momentum

$$I_G\,\omega_1 + \sum\int_{t_1}^{t_2} M_G\,dt = I_G\,\omega_2$$

For rotation about a fixed axis O we can also write:

$$I_O\,\omega_1 + \sum\int_{t_1}^{t_2} M_O\,dt = I_O\,\omega_2$$

17



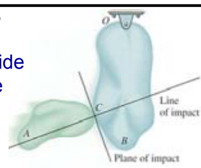
$$m(v_{Gx})_1 + \sum\int_{t_1}^{t_2} F_x dt = m(v_{Gx})_2$$

$$m(v_{Gy})_1 + \sum\int_{t_1}^{t_2} F_y dt = m(v_{Gy})_2$$

$$I_G\,\omega_1 + \sum\int_{t_1}^{t_2} M_G dt = I_G\,\omega_2$$

These equations can be applied to an entire system of connected bodies rather than to each body independently to eliminate the effect of reactions at the connections. 18

ECCENTRIC IMPACT



If two bodies with SMOOTH surfaces collide when their mass centers and velocity are aligned with the line of impact, the result is a central impact and is treated as we did before.

If the line connecting the mass centers does not coincide with the line of impact, as in the case when one of the bodies is rotating about a fixed axis the impact is ECCENTRIC.

In this case, in general two equations need to be solved to determine all the velocities. The first will generally involve the application of conservation of angular momentum and the second is obtained from the coefficient of restitution.

$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2 \quad \text{and} \quad e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$$

Where the velocities are along the line of impact.

19

EXAMPLES

1. A flywheel has a mass of 60 kg and a radius of gyration of $k_G = 150\,mm$ about an axis of rotation pasing through its mass center. If a motor supplies a clockwise torque having a magnitude of $M = (5t)\,N \cdot m$, where t is in seconds, determine the wheel's angular velocity in $t = 3s$. Initially the flywheel is rotating clockwise at $\omega_1 = 2\,rad/s$.

Data:

$m = 60\,kg$

$r_G = 0.15\,m$

$M = -5t$

$\omega_1 = -2\,rad/s$

$\omega(t = 3s) = ?$

SOLUTION

$$I_G = 60 \times 0.15^2 = 1.35\,kg \cdot m^2$$

$$(\mathbf{H}_0)_1 + \int_0^3 \mathbf{M}\,dt = (\mathbf{H}_0)_2$$

$$I_G\omega_1 + \int_0^3 (-5t)dt = I_G\omega_2$$

$$1.35 \times (-2) - \frac{5}{2}3^2 = 1.35\,\omega_2$$

$$\omega_2 = 18.67\,rad/s$$

20

2. The spool has a mass of 30 kg and a radius of gyration $k_O = 0.25m$. Block A has a mass of 25 kg, and block B has a mass of 10 kg. If they are released from rest, determine the time required for block A to attain speed of 2 m/s. Neglect the mass of the ropes.

Data:

$m_S = 30\,kg$

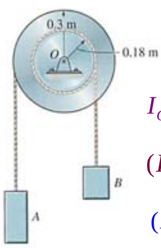
$k_O = 0.25\,m$

$m_A = 25\,kg$

$m_B = 10\,kg$

$v_A = 2\,m/s$

$t = ?$



SOLUTION

First calculate I_O

$$I_O = 30 \times 0.25^2 = 1.875\,kg \cdot m^2$$

$$(H_O)_1 + \sum \int_0^t M_O dt = (H_O)_2$$

$$(H_O)_1 = 0$$

21

$$\sum \int_0^t M_O dt = \int_0^t (m_A g\, r_A - m_B g\, r_B)\,dt = 9.81 \times (25 \times 0.3 - 10 \times 0.18)\,t$$

$$\sum \int_0^t M_O dt = 55.917\,t$$

$$(H_O)_2 = r_A\, m_A\, (v_A)_2 + r_B\, m_B\, (v_B)_2 + I_O\, \omega$$

(note that $\mathbf{r}_A \times \mathbf{v}_A$ and $\mathbf{r}_B \times \mathbf{v}_B$ have the same sign)

From $v_A = r_A\omega \;\Rightarrow\; \omega = \frac{2}{0.3} = 6.6667\,rad/s$

$v_B = r_B\omega \;\Rightarrow\; v_B = 0.18 \times 6.6667 = 1.2\,m/s$

Hence

$$(H_O)_2 = 0.3 \times 25 \times 2 + 0.18 \times 10 \times 1.2 + 1.875 \times 6.6667 = 29.66\,kg \cdot m^2/s$$

$$55.917\,t = 29.66$$

$$t = 0.53\,s$$

22