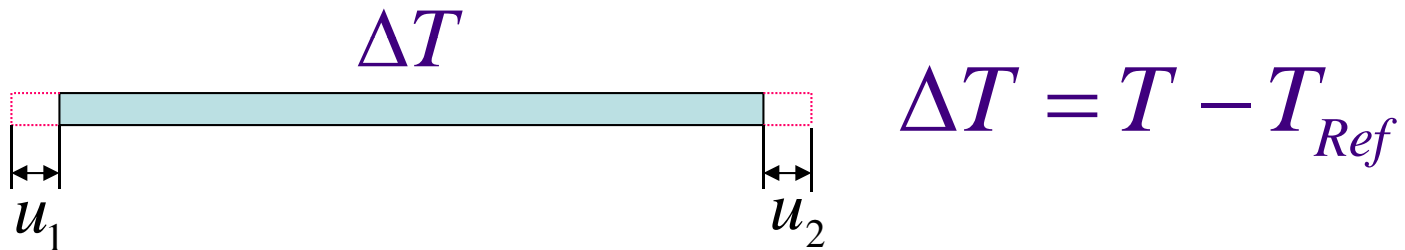


THERMAL STRESSES

Changes in temperature create significant stresses in all types of solid bodies, especially in aerospace and electronic components structures.

Consider a rod element. When subjected to a temperature change the rod will expand or contract



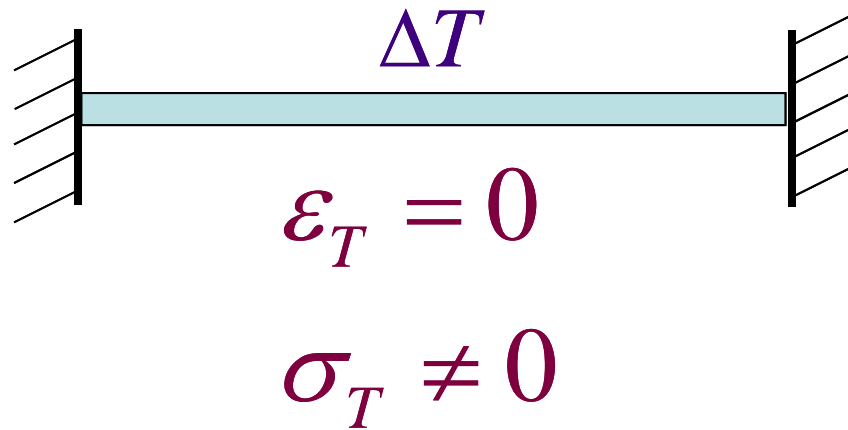
We will assume that the temperature field is known.

If the rod is not constrained and free to expand, a Thermal Strain ϵ_T will develop, but no Thermal Stress σ_T is produced.

$$\epsilon_T \neq 0$$

$$\sigma_T = 0$$

On the other hand if the rod is fully constrained so that it cannot Expand or contract, no Thermal Strain is possible and a Thermal Stress will develop.



This is a very different behavior from what we have seen so far when stresses are directly proportional to strains.

To describe the behavior of structures under thermal loading we define the **TOTAL STRAIN ϵ** as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_E + \boldsymbol{\varepsilon}_T$$

$\boldsymbol{\varepsilon}$ Is the Total Strain

$\boldsymbol{\varepsilon}_E$ Is the Elastic Strain

$\boldsymbol{\varepsilon}_T$ Is the Thermal Strain

Only the direct components of the Thermal Strains are non-zero, **temperature changes do not produce shear strains**. The direct Thermal Strains can be calculated directly from the temperature distribution and are:

$$\boldsymbol{\varepsilon}_T = \alpha \Delta T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} \boldsymbol{\varepsilon}_T = \alpha \Delta T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ \text{(2-D)} \end{matrix} \quad \begin{matrix} \boldsymbol{\varepsilon}_T = [\alpha \Delta T] \\ \text{(1-D)} \end{matrix}$$

α is the Coefficient of Thermal Expansion

The constitutive relation takes the form

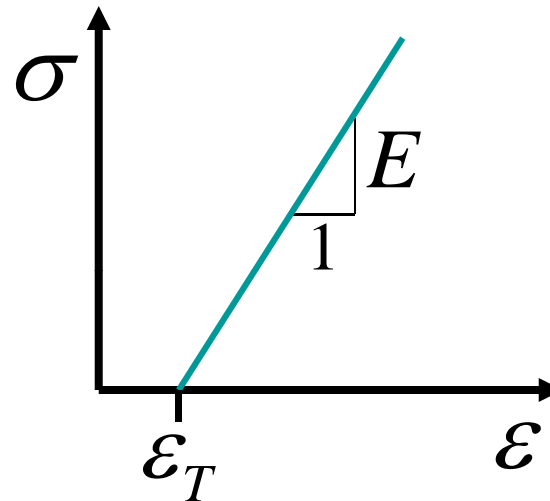
$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_T) \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_E + \boldsymbol{\varepsilon}_T$$

If the body is free to expand

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_T$$

and

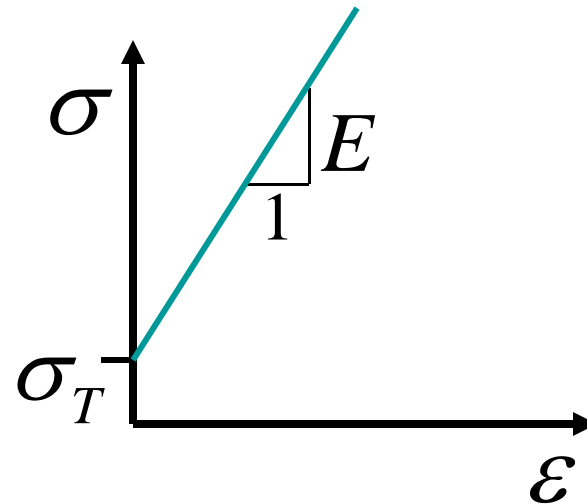
$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon}_T - \boldsymbol{\varepsilon}_T) = \mathbf{0}$$



$$(\sigma = E(\varepsilon - \varepsilon_T))$$

If the body is fully
Constrained

$$\boldsymbol{\varepsilon} = \mathbf{0} \text{ and } \boldsymbol{\sigma} = -\mathbf{D}\boldsymbol{\varepsilon}_T$$



POTENTIAL ENERGY

When Thermal Stresses are present only the Elastic component of the deformations do work, because when

$\boldsymbol{\varepsilon} \neq \mathbf{0}$ then $\boldsymbol{\sigma} = \mathbf{0}$ and when $\boldsymbol{\sigma} \neq \mathbf{0}$ then $\boldsymbol{\varepsilon} = \mathbf{0}$

The Total Strain Energy is: $F_{\varepsilon} = \frac{1}{2} \int_V (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_T)^T \mathbf{D} (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_T) dV$

Expanding:

$$F_{\varepsilon} = \frac{1}{2} \int_V \boldsymbol{\varepsilon}^T \mathbf{D} \boldsymbol{\varepsilon} dV - \int_V \boldsymbol{\varepsilon}^T \mathbf{D} \boldsymbol{\varepsilon}_T dV + \frac{1}{2} \int_V \boldsymbol{\varepsilon}_T^T \mathbf{D} \boldsymbol{\varepsilon}_T dV$$

Substituting $\boldsymbol{\varepsilon} = \mathbf{B} \mathbf{a}$

$$F_{\varepsilon} = \frac{1}{2} \mathbf{a}^T \left[\int_V \mathbf{B}^T \mathbf{D} \mathbf{B} dV \right] \mathbf{a} - \mathbf{a}^T \left[\int_V \mathbf{B}^T \mathbf{D} \boldsymbol{\varepsilon}_T dV \right] + \frac{1}{2} \left[\int_V \boldsymbol{\varepsilon}_T^T \mathbf{D} \boldsymbol{\varepsilon}_T dV \right]$$

After the minimization process

$$\left[\int_V \mathbf{B}^T \mathbf{D} \mathbf{B} dV \right] \mathbf{a} - \left[\int_V \mathbf{B}^T \mathbf{D} \boldsymbol{\varepsilon}_T dV \right] = 0$$

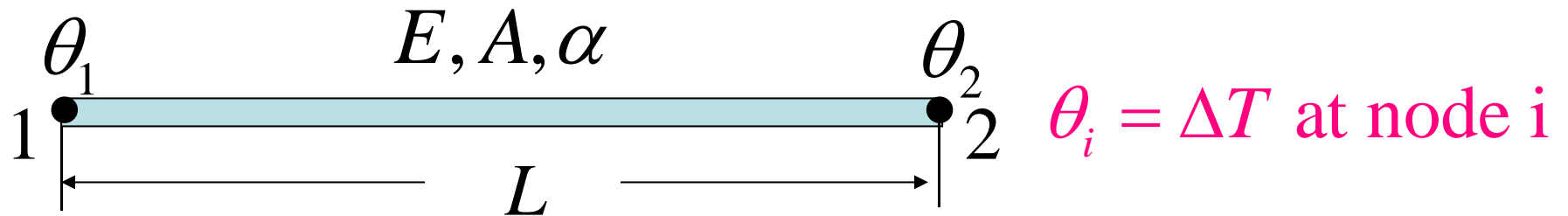
$$\mathbf{k} = \left[\int_V \mathbf{B}^T \mathbf{D} \mathbf{B} dV \right] \text{ as before}$$

And the Thermal Loading vector becomes

$$\mathbf{F}_T = \int_V \mathbf{B}^T \mathbf{D} \boldsymbol{\varepsilon}_T dV$$

which is known if the temperature field is known

For the Rod and Beam elements the Thermal Load vector is:



$$\mathbf{F}_T = \int_V \mathbf{B}^T \mathbf{D} \boldsymbol{\varepsilon}_T dV \quad \mathbf{B} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \quad \mathbf{D} = [\mathbf{E}] \quad \boldsymbol{\varepsilon}_T = \alpha \theta$$

$$\theta = \left(1 - \frac{x}{L}\right) \theta_1 + \frac{x}{L} \theta_2 \quad (\theta(x) = \mathbf{N} \boldsymbol{\theta})$$

$$\mathbf{F}_T = A \int_0^L \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} E \alpha \left(\left(1 - \frac{x}{L}\right) \theta_1 + \frac{x}{L} \theta_2 \right) dx$$

After integration: $\mathbf{F}_T = AE\alpha \left(\frac{\theta_1 + \theta_2}{2} \right) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Define $\bar{\theta} = \left(\frac{\theta_1 + \theta_2}{2} \right)$ The average temperature change

$$\mathbf{F}_T = AE\alpha \bar{\theta} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

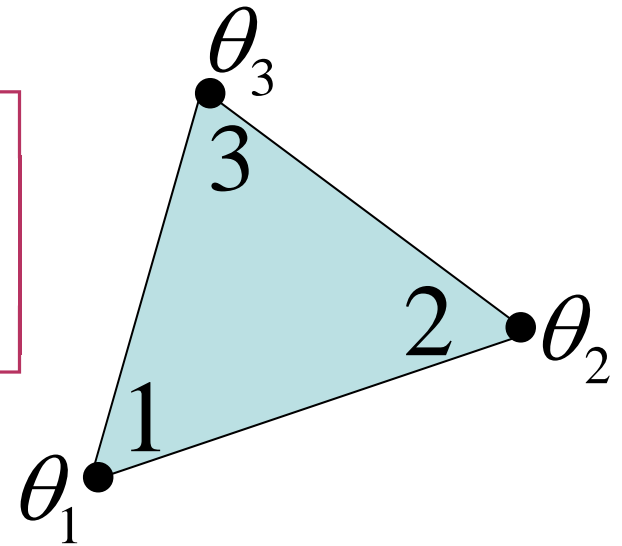
CONSTANT STRAIN TRIANGLE (plane stress)

$$\mathbf{B} = \frac{1}{2A} \begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix}$$

Use area coordinates

$$\theta = L_1\theta_1 + L_2\theta_2 + L_3\theta_3 = \mathbf{N}\boldsymbol{\theta} = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\varepsilon_T = \alpha \theta \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \alpha \mathbf{N} \boldsymbol{\theta} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$



$$\mathbf{F}_T = \int_V \mathbf{B}^T \mathbf{D} \alpha \mathbf{N} \boldsymbol{\theta} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} dV = t \alpha \mathbf{B}^T \mathbf{D} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \left(\int_A \mathbf{N} \boldsymbol{\theta} dA \right) \quad \mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$\mathbf{B}^T \mathbf{D} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \mathbf{B}^T \frac{E}{1-\nu^2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{E}{2A(1-\nu^2)} \begin{bmatrix} y_2 - y_3 \\ x_3 - x_2 \\ y_3 - y_1 \\ x_1 - x_3 \\ y_1 - y_2 \\ x_2 - x_1 \end{bmatrix}$$

$$\mathbf{B} = \frac{1}{2A} \begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix}$$

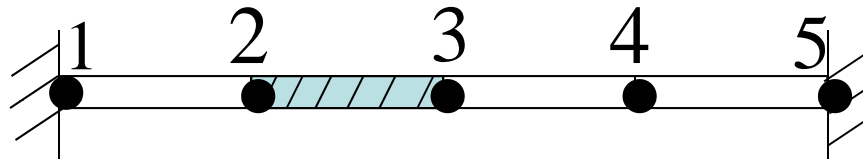
$$\int_A \mathbf{N} \boldsymbol{\theta} dA = \left(\int_A \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} dA \right) \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \frac{A}{3} (\theta_1 + \theta_2 + \theta_3) = A \bar{\theta}$$

Finally putting all together

$$\mathbf{F}_T = \frac{t \alpha E \bar{\theta}}{2(1-\nu)} \begin{bmatrix} y_2 - y_3 \\ x_3 - x_2 \\ y_3 - y_1 \\ x_1 - x_3 \\ y_1 - y_2 \\ x_2 - x_1 \end{bmatrix}$$

Example

The bar in the figure is modeled with 4 rod elements of equal length and has uniform properties A, E . Element 2-3 is heated up an average temperature $\bar{\theta}$. Find the stresses in the elements.



The global stiffness matrix is

$$\mathbf{K} = \frac{AE}{L} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

The thermal load vector is

$$\mathbf{F} = AE\alpha\bar{\theta} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

After applying boundary conditions the system of equations for displacements is

$$\frac{AE}{L} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = AE\alpha\bar{\theta} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Solution: $u_2 = -L\alpha\bar{\theta}/4$, $u_3 = L\alpha\bar{\theta}/2$, $u_4 = L\alpha\bar{\theta}/4$

STRESSES: $\sigma = E(\varepsilon - \varepsilon_T)$

Element 1-2: $\sigma = E \left(\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} 0 \\ -L\alpha\bar{\theta}/4 \end{bmatrix} - 0 \right) = -E\alpha\bar{\theta}/4$

no ε_T in element 1-2

Element 2-3: $\sigma = E \left(\left[-\frac{1}{L} \quad \frac{1}{L} \right] L \alpha \bar{\theta} / 4 \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \alpha \bar{\theta} \right) = -E \alpha \bar{\theta} / 4$

$\epsilon_T = \begin{bmatrix} \alpha \bar{\theta} \end{bmatrix} \epsilon_T$ in element 2-3

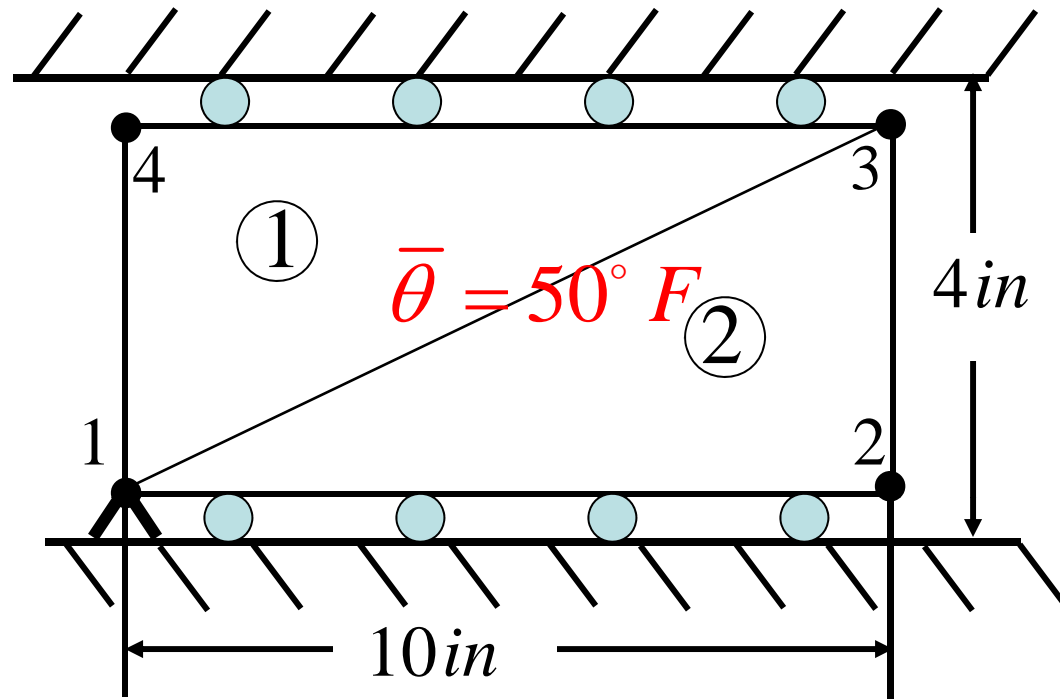
Element 3-4: $\sigma = E \left(\left[-\frac{1}{L} \quad \frac{1}{L} \right] L \alpha \bar{\theta} / 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 0 \right) = -E \alpha \bar{\theta} / 4$

Element 4-5: $\sigma = E \left(\left[-\frac{1}{L} \quad \frac{1}{L} \right] L \alpha \bar{\theta} / 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0 \right) = -E \alpha \bar{\theta} / 4$

Reactions: $R_1 = A E \alpha \bar{\theta} / 4$, $R_5 = -A E \alpha \bar{\theta} / 4$

Elements are in uniform compression. Element forces: $\mathbf{f} = \frac{A E \alpha \bar{\theta}}{4} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2-D EXAMPLE



$$E = 30 \times 10^6 \text{ psi}$$

$$t = 0.1 \text{ in}$$

$$\nu = 0.35$$

$$\alpha = 5 \times 10^{-6} 1/^\circ F$$

The thin plate in the figure is subjected to a temperature change of 50 degrees Fahrenheit. Find the stresses using two constant strain triangles as shown.

Use plane stress: $\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = 34.2 \times 10^6 \begin{bmatrix} 1 & 0.35 & 0 \\ 0.35 & 1 & 0 \\ 0 & 0 & 0.325 \end{bmatrix}$

For both elements $A = 20 \text{ in}^2$

Element 1: **Numbering the nodes: 3, 4, 1** $\mathbf{B}_1 = \frac{1}{20} \begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ 0 & 2 & 5 & -2 & -5 & 0 \end{bmatrix}$

$\mathbf{k}_1 = 1.71 \times 10^5$ $\begin{bmatrix} 4 & 0 & -4 & 3.5 & 0 & -3.5 \\ & 1.3 & 3.25 & -1.3 & -3.25 & 0 \\ & & 12.125 & -6.75 & -8.125 & 3.5 \\ & & & 26.3 & 3.25 & -25 \\ & \text{Symmetric} & & & 8.125 & 0 \\ & & & & & 25 \end{bmatrix}$

Element 2: **Numbering the nodes: 1, 2, 3**

$$\mathbf{B}_2 = \frac{1}{20} \begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \\ 0 & -2 & -5 & 2 & 5 & 0 \end{bmatrix}$$

Notice that $\mathbf{B}_2 = -\mathbf{B}_1$, hence $\mathbf{k}_2 = \mathbf{k}_1$

Right hand side vectors (thermal loads)

$$\mathbf{f}_1 = \frac{tE\alpha\bar{\theta}}{2(1-\nu)} \begin{bmatrix} y_2 - y_3 \\ x_3 - x_2 \\ y_3 - y_1 \\ x_1 - x_3 \\ y_1 - y_2 \\ x_2 - x_1 \end{bmatrix} = 577. \begin{bmatrix} 4 \\ 0 \\ -4 \\ 10 \\ 0 \\ -10 \end{bmatrix}$$

$$\mathbf{f}_2 = 577. \begin{bmatrix} -4 \\ 0 \\ 4 \\ -10 \\ 0 \\ 10 \end{bmatrix}$$

Assembled system

$$1.71 \times 10^5 \begin{bmatrix} 12.125 & 0 & -4 & 3.5 & 0 & -6.75 & -8.125 & 3.25 \\ & 26.3 & 3.25 & -1.3 & -6.75 & 0 & 3.5 & -25 \\ & & 12.125 & -6.75 & -8.125 & 3.5 & 0 & 0 \\ & & & 26.3 & 3.25 & -25 & 0 & 0 \\ & & & & 12.125 & 0 & -4 & 3.5 \\ & \text{Symmetric} & & & & 26.3 & 3.25 & -1.3 \\ & & & & & & 12.125 & -6.75 \\ & & & & & & & 26.3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = 577. \begin{bmatrix} -4 \\ -10 \\ 4 \\ -10 \\ 4 \\ 10 \\ -4 \\ 10 \end{bmatrix}$$

Apply boundary conditions

$$u_1 = v_1 = v_2 = v_3 = v_4 = 0$$

$$1.71 \times 10^5 \begin{bmatrix} 12.125 & -8.125 & 0 \\ -8.125 & 12.125 & -4 \\ 0 & -4 & 12.125 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = 577. \begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix}$$

Solution: $u_2 = 3.37 \times 10^{-3} \text{ in}$, $u_3 = 3.37 \times 10^{-3} \text{ in}$, $u_4 = 0.0 \text{ in}$

Stresses: $\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_T)$ $\boldsymbol{\varepsilon}_T = \alpha \bar{\theta} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2.5 \times 10^{-4} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Element1: $\boldsymbol{\varepsilon}_1 = \mathbf{B}_1 \mathbf{a}_1 = \frac{1}{20} \begin{bmatrix} 2 & -2 \\ 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = 3.37 \times 10^{-4} \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix}$

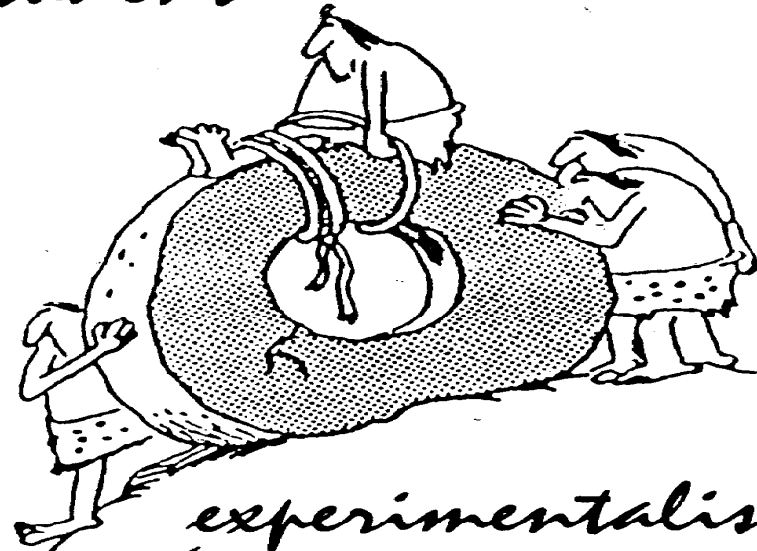
$$\sigma_1 = \mathbf{D}(\epsilon_1 - \epsilon_T) = 34.2 \times 10^6 \begin{bmatrix} 1 & .35 & 0 \\ .35 & 1 & 0 \\ 0 & 0 & .325 \end{bmatrix} 10^{-4} \left(\begin{bmatrix} 3.37 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 2.5 \\ 0 \end{bmatrix} \right)$$

$$\sigma_1 = \begin{bmatrix} 0 \\ -7.5 \text{ kpsi} \\ 0 \end{bmatrix}$$

$$\text{Element 2: } \epsilon_2 = \frac{1}{20} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ -5 & 5 \end{bmatrix} 10^{-3} \begin{bmatrix} 3.37 \\ 3.37 \end{bmatrix} = 3.37 \times 10^{-4} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

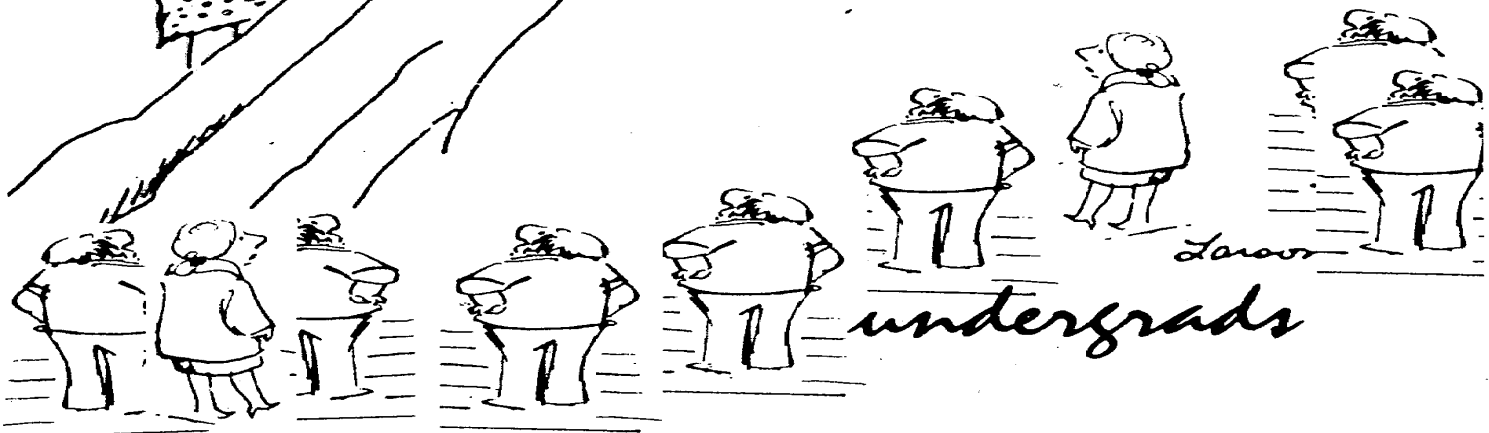
$$\sigma_2 = \begin{bmatrix} 0 \\ -7.5 \text{ kpsi} \\ 0 \end{bmatrix}$$

graduate student



experimentalists

theorist



undergrads