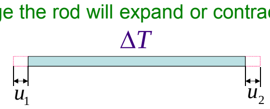
THERMAL STRESSES

Changes in temperature create significant stresses in all types of solid bodies, especially in aerospace and electronic components structures.

Consider a rod element. When subjected to a temperature change the rod will expand or contract



$$\Delta T = T - T_{Ref}$$

We will assume that the temperature field is known.
If the rod is not constrained and free to expand, a Thermal Strain ε_T will develop, but no Thermal Stress σ_T is produced.

$$\varepsilon_T \neq 0$$

$$\sigma_T = 0$$

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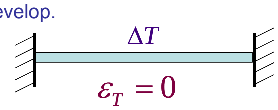
On the other hand if the rod is fully constrained so that it cannot Expand or contract, no Thermal Strain is possible and a Thermal Stress will develop.



$$\varepsilon_T = 0$$

$$\sigma_T \neq 0$$

This is a very different behavior from what we have seen so far when stresses are directly proportional to strains.

To describe the behavior of structures under thermal loading we define the **TOTAL STRAIN** ε as

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$$\varepsilon = \varepsilon_E + \varepsilon_T$$

ε Is the Total Strain

ε_E Is the Elastic Strain

ε_T Is the Thermal Strain

Only the direct components of the Thermal Strains are non-zero, temperature changes do not produce shear strains. The direct Thermal Strains can be calculated directly from the temperature distribution and are:

$$\varepsilon_T = \alpha\,\Delta T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \varepsilon_T = \alpha\,\Delta T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ (2-D)} \qquad \varepsilon_T = [\alpha\,\Delta T] \text{ (1-D)}$$

α is the Coefficient of Thermal Expansion

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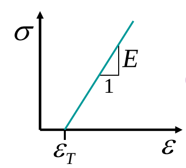
The constitutive relation takes the form

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_T) \qquad \varepsilon = \varepsilon_E + \varepsilon_T$$

If the body is free to expand

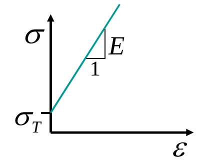
$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_T$ and $\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon}_T - \boldsymbol{\varepsilon}_T) = 0$



$$(\sigma = E(\varepsilon - \varepsilon_T))$$

If the body is fully Constrained

$\boldsymbol{\varepsilon} = 0$ and $\boldsymbol{\sigma} = -\mathbf{D}\boldsymbol{\varepsilon}_T$



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POTENTIAL ENERGY

When Thermal Stresses are present only the Elastic component of the deformations do work, because when

$$\boldsymbol{\varepsilon} \neq 0 \text{ then } \boldsymbol{\sigma} = 0 \text{ and when } \boldsymbol{\sigma} \neq 0 \text{ then } \boldsymbol{\varepsilon} = 0$$

The Total Strain Energy is: $F_\varepsilon = \frac{1}{2}\int_V (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_T)^T \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_T)\,dV$

Expanding:

$$F_\varepsilon = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^T\mathbf{D}\boldsymbol{\varepsilon}\,dV - \int_V \boldsymbol{\varepsilon}^T\mathbf{D}\boldsymbol{\varepsilon}_T\,dV + \frac{1}{2}\int_V \boldsymbol{\varepsilon}_T^T\mathbf{D}\boldsymbol{\varepsilon}_T\,dV$$

Substituting $\boldsymbol{\varepsilon} = \mathbf{Ba}$

$$F_\varepsilon = \frac{1}{2}\mathbf{a}^T\left[\int_V \mathbf{B}^T\mathbf{DB}\,dV\right]\mathbf{a} - \mathbf{a}^T\left[\int_V \mathbf{B}^T\mathbf{D}\boldsymbol{\varepsilon}_T\,dV\right] + \frac{1}{2}\left[\int_V \boldsymbol{\varepsilon}_T^T\mathbf{D}\boldsymbol{\varepsilon}_T\,dV\right]$$

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After the minimization process

$$\left[\int_V \mathbf{B}^T\mathbf{DB}\,dV\right]\mathbf{a} - \left[\int_V \mathbf{B}^T\mathbf{D}\boldsymbol{\varepsilon}_T\,dV\right] = 0$$

$$\mathbf{k} = \left[\int_V \mathbf{B}^T\mathbf{DB}\,dV\right] \quad \text{as before}$$

And the Thermal Loading vector becomes

$$\mathbf{F}_T = \int_V \mathbf{B}^T\mathbf{D}\boldsymbol{\varepsilon}_T\,dV$$

which is known if the temperature field is known

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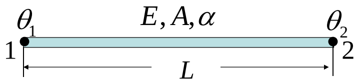
For the Rod and Beam elements the Thermal Load vector is:



$\theta_i = \Delta T$ at node i

$$\mathbf{F_T} = \int_V \mathbf{B^T D} \varepsilon_\mathbf{T}\, dV \qquad \mathbf{B} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \qquad \mathbf{D} = [\mathbf{E}] \qquad \varepsilon_T = \alpha\theta$$

$$\theta = \left(1 - \frac{x}{L}\right)\theta_1 + \frac{x}{L}\theta_2 \qquad \left(\theta(x) = \mathbf{N\theta}\right)$$

$$\mathbf{F_T} = A\int_0^L \begin{bmatrix} -\dfrac{1}{L} \\ \dfrac{1}{L} \end{bmatrix} E\alpha\left((1 - \frac{x}{L})\theta_1 + \frac{x}{L}\theta_2\right) dx$$

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After integration: $\mathbf{F_T} = AE\alpha\left(\dfrac{\theta_1 + \theta_2}{2}\right)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Define $\bar{\theta} = \left(\dfrac{\theta_1 + \theta_2}{2}\right)$ The average temperature change

$$\mathbf{F_T} = AE\alpha\bar{\theta}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

CONSTANT STRAIN TRIANGLE (plane stress)

$$\mathbf{B} = \frac{1}{2A}\begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix}$$

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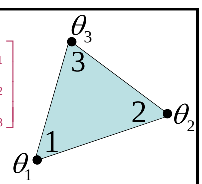
Use area coordinates



$$\theta = L_1\theta_1 + L_2\theta_2 + L_3\theta_3 = \mathbf{N\theta} = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\varepsilon_T = \alpha\,\theta\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \alpha\,\mathbf{N}\,\boldsymbol{\theta}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{F_T} = \int_V \mathbf{B^T D}\alpha\mathbf{N\theta}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} dV = t\alpha B^T D\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\left(\int_A \mathbf{N\theta}\, dA\right) \qquad D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$\mathbf{B^T D}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \mathbf{B^T}\frac{E}{1-\nu^2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{E}{2A(1-\nu^2)}\begin{bmatrix} y_2 - y_3 \\ x_3 - x_2 \\ y_3 - y_1 \\ x_1 - x_3 \\ y_1 - y_2 \\ x_2 - x_1 \end{bmatrix}$$

$$\mathbf{B} = \frac{1}{2A}\begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix}$$

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$$\int_A \mathbf{N\theta}\, dA = \left(\int_A \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} dA\right)\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \frac{A}{3}(\theta_1 + \theta_2 + \theta_3) = A\bar{\theta}$$

Finally putting all together

$$\mathbf{F_T} = \frac{t\alpha E\bar{\theta}}{2(1-\nu)}\begin{bmatrix} y_2 - y_3 \\ x_3 - x_2 \\ y_3 - y_1 \\ x_1 - x_3 \\ y_1 - y_2 \\ x_2 - x_1 \end{bmatrix}$$

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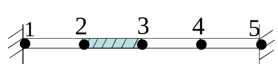
Example

The bar in the figure is modeled with 4 rod elements Of equal length and has uniform properties A,E . Element 2-3 is heated up an average temperature $\bar{\theta}$. Find the stresses in the elements.



The global stiffness matrix is

$$\mathbf{K} = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

The thermal load vector is

$$\mathbf{F} = AE\alpha\bar{\theta}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

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After applying boundary conditions the system of equations for displacements is

$$\frac{AE}{L}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = AE\alpha\bar{\theta}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Solution: $u_2 = -L\alpha\bar{\theta}/4,\ u_3 = L\alpha\bar{\theta}/2,\ u_4 = L\alpha\bar{\theta}/4$

STRESSES: $\sigma = E(\varepsilon - \varepsilon_T)$

Element 1-2: $\sigma = E\left(\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{bmatrix} 0 \\ -L\alpha\bar{\theta}/4 \end{bmatrix} - 0\right) = -E\alpha\bar{\theta}/4$

no ε_T in element 1-2

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Element 2-3: $\sigma = E\left(\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} L\alpha\bar{\theta}/4 \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \alpha\bar{\theta}\right) = -E\alpha\bar{\theta}/4$

$\boldsymbol{\varepsilon}_T = \begin{bmatrix} \alpha\bar{\theta} \end{bmatrix}$ ε_T **in element 2-3**

Element 3-4: $\sigma = E\left(\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} L\alpha\bar{\theta}/4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 0\right) = -E\alpha\bar{\theta}/4$

Element 4-5: $\sigma = E\left(\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} L\alpha\bar{\theta}/4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0\right) = -E\alpha\bar{\theta}/4$

Reactions: $R_1 = AE\alpha\bar{\theta}/4$, $R_5 = -AE\alpha\bar{\theta}/4$

Elements are in uniform compression. Element forces: $\mathbf{f} = \frac{AE\alpha\bar{\theta}}{4}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

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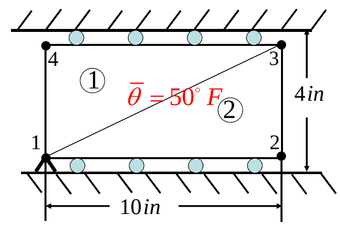
2-D EXAMPLE



$E = 30\times10^6\ psi$
$t = 0.1\ in$
$\nu = 0.35$
$\alpha = 5\times10^{-6}\ 1/^\circ F$

The thin plate in the figure is subjected to a temperature change of 50 degrees Fahrenheit. Find the stresses using two constant strain triangles as shown.

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Use plane stress: $\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = 34.2\times10^6\begin{bmatrix} 1 & 0.35 & 0 \\ 0.35 & 1 & 0 \\ 0 & 0 & 0.325 \end{bmatrix}$

For both elements $A = 20\ in^2$

Element 1: Numbering the nodes: 3, 4, 1 $\mathbf{B}_1 = \frac{1}{20}\begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ 0 & 2 & 5 & -2 & -5 & 0 \end{bmatrix}$

$$\mathbf{k}_1 = 1.71\times10^5\begin{bmatrix} 4 & 0 & -4 & 3.5 & 0 & -3.5 \\ & 1.3 & 3.25 & -1.3 & -3.25 & 0 \\ & & 12.125 & -6.75 & -8.125 & 3.5 \\ & & & 26.3 & 3.25 & -25 \\ & \text{Symmetric} & & & 8.125 & 0 \\ & & & & & 25 \end{bmatrix}$$

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Element 2: Numbering the nodes: 1, 2, 3 $\mathbf{B}_2 = \frac{1}{20}\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \\ 0 & -2 & -5 & 2 & 5 & 0 \end{bmatrix}$

Notice that $\mathbf{B}_2 = -\mathbf{B}_1$ **, hence** $\mathbf{k}_2 = \mathbf{k}_1$

Right hand side vectors (thermal loads)

$$\mathbf{f}_1 = \frac{tE\alpha\bar{\theta}}{2(1-\nu)}\begin{bmatrix} y_2 - y_3 \\ x_3 - x_2 \\ y_3 - y_1 \\ x_1 - x_3 \\ y_1 - y_2 \\ x_2 - x_1 \end{bmatrix} = 577.\begin{bmatrix} 4 \\ 0 \\ -4 \\ 10 \\ 0 \\ -10 \end{bmatrix} \qquad \mathbf{f}_2 = 577.\begin{bmatrix} -4 \\ 0 \\ 4 \\ -10 \\ 0 \\ 10 \end{bmatrix}$$

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Assembled system

$$1.71\times10^5\begin{bmatrix} 12.125 & 0 & -4 & 3.5 & 0 & -6.75 & -8.125 & 3.25 \\ & 26.3 & 3.25 & -1.3 & -6.75 & 0 & 3.5 & -25 \\ & & 12.125 & -6.75 & -8.125 & 3.5 & 0 & 0 \\ & & & 26.3 & 3.25 & -25 & 0 & 0 \\ & & & & 12.125 & 0 & -4 & 3.5 \\ & \text{Symmetric} & & & & 26.3 & 3.25 & -1.3 \\ & & & & & & 12.125 & -6.75 \\ & & & & & & & 26.3 \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = 577.\begin{bmatrix} -4 \\ -10 \\ 4 \\ -10 \\ 4 \\ 10 \\ -4 \\ 10 \end{bmatrix}$$

Apply boundary conditions

$$u_1 = v_1 = v_2 = v_3 = v_4 = 0$$

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$$1.71\times10^5\begin{bmatrix} 12.125 & -8.125 & 0 \\ -8.125 & 12.125 & -4 \\ 0 & -4 & 12.125 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = 577.\begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix}$$

Solution: $u_2 = 3.37\times10^{-3} in$, $u_3 = 3.37\times10^{-3} in$, $u_4 = 0.0 in$

Stresses: $\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_T)$ $\boldsymbol{\varepsilon}_T = \alpha\bar{\theta}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2.5\times10^{-4}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Element1: $\boldsymbol{\varepsilon}_1 = \mathbf{B}_1\mathbf{a}_1 = \frac{1}{20}\begin{bmatrix} 2 & -2 \\ 0 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = 3.37\times10^{-4}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

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$$\boldsymbol{\sigma}_1 = \mathbf{D}(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_T) = 34.2\times10^6 \begin{bmatrix} 1 & .35 & 0 \\ .35 & 1 & 0 \\ 0 & 0 & .325 \end{bmatrix} 10^{-4} \left(\begin{bmatrix} 3.37 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 2.5 \\ 0 \end{bmatrix} \right)$$

$$\boldsymbol{\sigma}_1 = \begin{bmatrix} 0 \\ -7.5\,kpsi \\ 0 \end{bmatrix}$$

Element 2: $\boldsymbol{\varepsilon}_2 = \frac{1}{20} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ -5 & 5 \end{bmatrix} 10^{-3} \begin{bmatrix} 3.37 \\ 3.37 \end{bmatrix} = 3.37\times10^{-4} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\boldsymbol{\sigma}_2 = \begin{bmatrix} 0 \\ -7.5\,kpsi \\ 0 \end{bmatrix}$$

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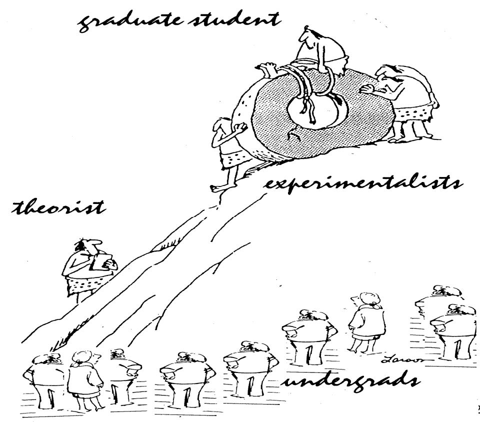

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