ME 360L: Solution to Exam 2

1. a)

$$x_1 = 0.03m \qquad y_1 = 0.0$$
$$x_2 = 0.0 \qquad y_2 = 0.02m \qquad 2A = 0.0006m^2$$
$$x_3 = 0.0 \qquad y_3 = 0.0$$

The matrix **B** is

$$\mathbf{B} = \frac{1}{2A}\begin{bmatrix} y_2-y_3 & 0 & y_3-y_1 & 0 & y_1-y_2 & 0 \\ 0 & x_3-x_2 & 0 & x_1-x_3 & 0 & x_2-x_1 \\ x_3-x_2 & y_2-y_3 & x_1-x_3 & y_3-y_1 & x_2-x_1 & y_1-y_2 \end{bmatrix} = \frac{1}{0.0006}\begin{bmatrix} 0.02 & 0 & 0 & 0 & -0.02 & 0 \\ 0 & 0 & 0 & 0.03 & 0 & -0.03 \\ 0 & 0.02 & 0.03 & 0 & -0.03 & -0.02 \end{bmatrix}$$

b) The constitutive matrix **D** is the matrix for plane stress

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} = \frac{7\times10^{10}}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} = 7.6923\times10^{10}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

The matrix **DB** becomes

$$\mathbf{DB} = 7.6923\times10^{10}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} 16.667 \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 & 0 & -3 \\ 0 & 2 & 3 & 0 & -3 & -2 \end{bmatrix}$$

$$\mathbf{DB} = 128.2\times10^{10}\begin{bmatrix} 2 & 0 & 0 & 0.9 & -2 & -0.9 \\ 0.6 & 0 & 0 & 3 & -0.6 & -3 \\ 0 & 0.7 & 1.05 & 0 & -1.05 & -0.7 \end{bmatrix}$$

And the element stiffness matrix **K** becomes

$$\mathbf{K} = tA\mathbf{B}^{\mathrm{T}}\mathbf{DB} = \frac{0.002\times10^{-2}}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \\ -2 & 0 & -3 \\ 0 & -3 & -2 \end{bmatrix} 128.2\times10^{10}\begin{bmatrix} 2 & 0 & 0 & 0.9 & -2 & -0.9 \\ 0.6 & 0 & 0 & 3 & -0.6 & -3 \\ 0 & 0.7 & 1.05 & 0 & -1.05 & -0.7 \end{bmatrix}$$

$$\mathbf{K} = 1.282\times10^{7}\begin{bmatrix} 4 & 0 & 0 & 1.8 & -4 & -1.8 \\ 0 & 1.4 & 2.1 & 0 & -2.1 & -1.4 \\ 0 & 2.1 & 3.15 & 0 & -3.15 & -2.1 \\ 1.8 & 0 & 0 & 9 & -1.8 & -9 \\ -4 & -2.1 & -3.15 & -1.8 & 7.15 & 3.9 \\ -1.8 & -1.4 & -2.1 & -9 & 3.9 & 10.4 \end{bmatrix}$$

c) The displacements at node 1 are

$$1.282\times10^{7}\begin{bmatrix} 4 & 0 \\ 0 & 1.4 \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -10^4 \end{bmatrix} \Rightarrow \begin{bmatrix} u_1 = 0.0 \\ v_1 = -5.57\times10^{-4}\,m \end{bmatrix} = \begin{bmatrix} 0 \\ -0.557\,mm \end{bmatrix}$$

d) The stresses in the element are

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \mathbf{DBa} = 128.2\times10^{10}\begin{bmatrix} 2 & 0 & 0 & 0.9 & -2 & -0.9 \\ 0.6 & 0 & 0 & 3 & -0.6 & -3 \\ 0 & 0.7 & 1.05 & 0 & -1.05 & -0.7 \end{bmatrix}\begin{bmatrix} 0 \\ -5.57\times10^{-4} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -5\times10^{8} \end{bmatrix} Pa$$

2. a) $\mathbf{k}_1 = \mathbf{k}_3 = \dfrac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $\qquad \mathbf{k}_2 = \dfrac{2AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

b) $\mathbf{f}_{1-2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\qquad \mathbf{f}_{2-3} = 2EA\alpha\theta\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\qquad \mathbf{f}_{3-4} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

c)

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = 2EA\alpha\theta\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = 2L\alpha\theta\begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \frac{2L\alpha\theta}{5}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

d)

$$\sigma_{1-2} = \frac{E}{L}(u_2 - u_1) = \frac{E}{L}\left(-\frac{2L\alpha\theta}{5}\right) = -\frac{2E\alpha\theta}{5}$$

$$\sigma_{2-3} = \frac{E}{L}(u_3 - u_2) - E\alpha\theta = \frac{4E\alpha\theta}{5} - E\alpha\theta = -\frac{E\alpha\theta}{5}$$

$$\sigma_{3-4} = \frac{E}{L}(u_4 - u_3) = \frac{E}{L}\left(-\frac{2L\alpha\theta}{5}\right) = -\frac{2E\alpha\theta}{5}$$