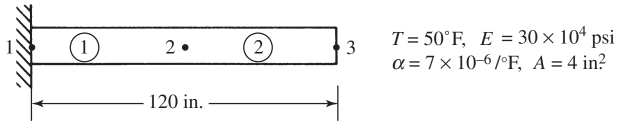
Chapter 15

15.1



$$[k^{(1)}] = \frac{AE}{60}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (1,\ 2), \qquad [k^{(2)}] = \frac{AE}{60}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (2,\ 3)$$

$$\{f^{(1)}\} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}, \qquad \{f^{(2)}\} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}$$

$\{F\} = [K]\{d\}$ becomes

$$\begin{Bmatrix} -E\alpha TA \\ 0 \\ E\alpha TA \end{Bmatrix} = \frac{AE}{60}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

Solving

$$u_2 = \alpha T L$$
$$= (7 \times 10^{-6})\ (50°\text{F})\ (60\text{ in.})$$
$$= 0.021\text{ in.}$$

$$u_3 = 2\ \alpha T L = 0.042\text{ in.}$$

Reactions and actual nodal forces

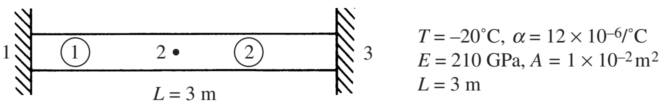
$$\{F\} = [K]\{d\} - \{F_0\}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\left(\frac{AE}{L}\right)\begin{Bmatrix} 0 \\ \alpha TL \\ 2\alpha TL \end{Bmatrix} - \begin{Bmatrix} -E\alpha TA \\ 0 \\ E\alpha TA \end{Bmatrix}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^{(1)} = \sigma^{(2)} = \frac{0}{4\text{ in.}^2} = 0$$

15.2



$$[k^{(1)}] = \frac{AE}{1.5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [k^{(2)}] = \frac{AE}{1.5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{f^{(1)}\} = \{f^{(2)}\} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}$$

Global equations

$$\frac{AE}{1.5}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \end{Bmatrix} = \begin{Bmatrix} -E\alpha TA \\ 0 \\ E\alpha TA \end{Bmatrix}$$

Solving

$$\frac{AE}{1.5}\,(2u_2) = 0$$

$$u_2 = 0$$

Forces in elements

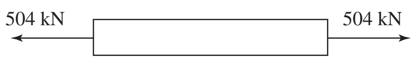
$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix} = \begin{Bmatrix} E\alpha TA \\ -E\alpha TA \end{Bmatrix}$$

$$E\,\alpha\,T\,A = (210 \text{ GPa})\,(12 \times 10^{-6}/°\text{C})\,(-20°\text{C}) \times (1 \times 10^{-2} \text{ m}^2)$$

$$= -504 \text{ kN}$$

$$\therefore f_{1x}^{(1)} = -504 \text{ kN},\ f_{2x}^{(1)} = 504 \text{ kN}$$

FBD element 1



$$\sigma^{(1)} = \frac{504 \text{ kN}}{1\times 10^{-2}\text{ m}^2} = 50{,}400 \text{ KPa}$$

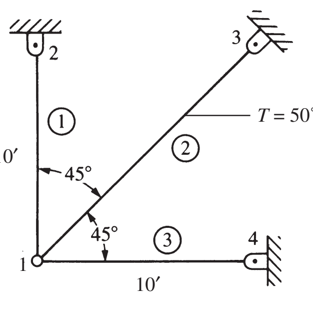
$$= 50.4 \text{ MPa}$$

Similarly

$$\sigma^{(2)} = 50.4 \text{ MPa}$$

$$F_{1x} = -504 \text{ kN},\ F_{3x} = 504 \text{ kN}$$

15.3



$\alpha = 7 \times 10^{-6}/°\text{F}$, $E = 30 \times 10^6$ psi
$A = 2$ in.^{2}, $f_{1x} = -E\alpha TA$

$$\{f\} = [T]^T\,\{f'\}$$

$$\begin{Bmatrix} f_{1x}'^{(2)} \\ f_{3x}'^{(2)} \end{Bmatrix} = \begin{Bmatrix} -21000 \\ 21000 \end{Bmatrix}$$

$$\{f\} = [T]^T\,\{f'\}$$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{bmatrix} 0.707 & -0.707 & 0 & 0 \\ 0.707 & 0.707 & 0 & 0 \\ 0 & 0 & 0.707 & -0.707 \\ 0 & 0 & 0.707 & 0.707 \end{bmatrix} \begin{Bmatrix} f'_{1x} = -21{,}000 \\ 0 \\ 21{,}000 \\ 0 \end{Bmatrix}$$

$$f_{1x} = -14{,}850 \text{ lb}, \quad f_{1y} = -14{,}850 \text{ lb}$$

$$f_{3x} = 14{,}850 \text{ lb}, \quad f_{3y} = -14{,}850 \text{ lb}$$

Boundary conditions

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

$$\begin{Bmatrix} F_{1x} = -14{,}850 \\ F_{1y} = 14{,}850 \end{Bmatrix} = 500000 \begin{bmatrix} 1.354 & 0.354 \\ 0.354 & 1.354 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

Solving

$$-u_1 = v_1 = -0.01753 \text{ in.}$$

By Equation (14.1.57)

$$\sigma^{(1)} = \frac{E}{L} [-C \quad -S \quad C \quad S] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} - \{0\}$$

$$\sigma^{(1)} = \frac{30 \times 10^6}{120 \text{ in.}} [0 \ -1 \ 0 \ 1] \begin{Bmatrix} 0.01753 \\ -0.01753 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^{(1)} = 4350 \text{ psi (T)}$$

$$\sigma^{(2)} = \frac{E}{L} [-C \quad -S \quad C \quad S] \begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{Bmatrix} - E\alpha T$$

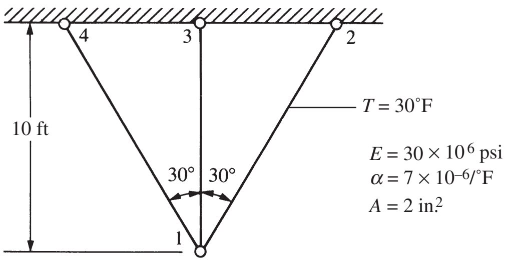
$$= \frac{30 \times 10^6}{120\sqrt{2}} [-0.707 \ -0.707 \ 0.707 \ 0.707] \begin{Bmatrix} 0.01753 \\ -0.01753 \\ 0 \\ 0 \end{Bmatrix}$$

$$= -10500$$

$$\sigma^{(2)} = -6150 \text{ psi (C)}$$

$$\sigma^{(3)} = 4350 \text{ psi (T)}$$

15.4



$$\{f'^{(1)}\} = \begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix} = \begin{Bmatrix} -12600 \\ 12600 \end{Bmatrix} \text{lb}$$

$$\{f\} = [T]^T \{f'\}$$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} f'_{1x} = -12600 \\ f'_{1y} = 0 \\ f'_{2x} = 12600 \\ f'_{2y} = 0 \end{Bmatrix}$$

$$f_{1x} = -6300 \text{ lb}$$
$$f_{1y} = -10912 \text{ lb}$$
$$f_{2x} = 6300 \text{ lb}$$
$$f_{2y} = 10912 \text{ lb}$$

Boundary conditions

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

Global equations

$$\begin{Bmatrix} F_{1x} = -6300 \\ F_{1y} = -10912 \end{Bmatrix} = \frac{(2 \text{ in.}^2)(30 \times 10^6)}{120 \text{ in.}} \begin{bmatrix} \frac{\sqrt{3}}{4} & 0 \\ 0 & 1 + \frac{3\sqrt{3}}{4} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$-6300 = 216{,}506\, u_1 \qquad u_1 = -0.0291 \text{ in.}$$
$$-10912 = 1{,}149{,}519\, v_1 \qquad v_1 = -0.0095 \text{ in.}$$

$$\sigma^{(1)} = \frac{E}{L} [-C \quad -S \quad C \quad S] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} - E\alpha T$$

$$= \frac{30 \times 10^6}{\frac{2(120)}{\sqrt{3}}} \left[-\frac{1}{2} \quad \frac{-\sqrt{3}}{2} \quad \frac{1}{2} \quad \frac{\sqrt{3}}{2} \right] \begin{Bmatrix} -0.0291 \\ -0.0095 \\ 0 \\ 0 \end{Bmatrix}$$

$$- 30 \times 10^6 \times 7 \times 10^{-6} \times 30°\text{F}$$

$$\sigma^{(1)} = (216506)\,(0.0228) - 6300$$
$$= -1370 \text{ psi (C)}$$

$$\sigma^{(2)} = \frac{30 \times 10^6}{120} [0 \quad -1 \quad 0 \quad 1] \begin{Bmatrix} -0.0291 \\ -0.0095 \\ 0 \\ 0 \end{Bmatrix} - 0$$

$$\sigma^{(2)} = 2375 \text{ psi (T)}$$

$$\sigma^{(3)} = \frac{30 \times 10^6}{\frac{2(120)}{\sqrt{3}}} \left[\frac{1}{2} \quad -\frac{\sqrt{3}}{2} \quad -\frac{1}{2} \quad \frac{\sqrt{3}}{2} \right] \begin{Bmatrix} -0.0291 \\ -0.0095 \\ 0 \\ 0 \end{Bmatrix} - 0$$

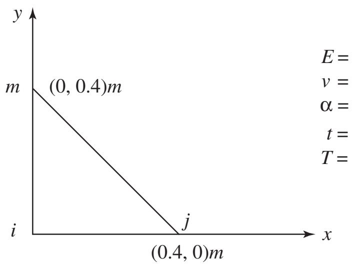
$$\sigma^{(3)} = -1370 \text{ psi (C)}$$

$$\beta_m = y_i - y_j = 0 - 0 = 0, \qquad \gamma_m = x_j - x_i = 6 - 0 = 6$$

$$\{f_T\} = \frac{(7.0\times10^{-6})(30\times10^{6})(1)(100°\text{F})}{2(1-0.3)}\begin{Bmatrix} -4 \\ -6 \\ 4 \\ 0 \\ 0 \\ 6 \end{Bmatrix}$$

$$= \begin{Bmatrix} -60,000 \\ -90,000 \\ 60,000 \\ 0 \\ 0 \\ 90,000 \text{ lb} \end{Bmatrix}$$

15.16



$$\{f_T\} = \frac{\alpha E t T}{2(1-v)}\begin{Bmatrix} \beta_i \\ \gamma_i \\ \beta_j \\ \gamma_j \\ \beta_m \\ \gamma_m \end{Bmatrix}$$

$\beta_i = -0.4$ m $\qquad \gamma_i = -0.4$ m

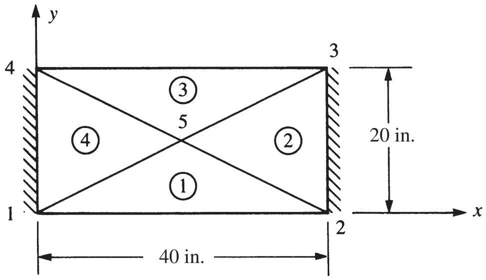
$\beta_j = 0.4$ m $\qquad \gamma_j = 0$

$\beta_m = 0 \qquad \gamma_m = 0.4$ m

$$\{f_T\} = \frac{(12\times10^{-6})(210\times10^{9})(0.01)(-20°\text{C})}{2(1-0.25)}\begin{Bmatrix} -0.4 \\ -0.4 \\ 0.4 \\ 0 \\ 0 \\ 0.4 \end{Bmatrix}$$

$$\{f_T\} = \begin{Bmatrix} 134.4 \\ 134.4 \\ -134.4 \\ 0 \\ 0 \\ -134.4 \end{Bmatrix} \text{kN}$$

15.17



Thermal force matrix

Element 1

$$i = 1, j = 2, m = 5$$

$$\beta_i = y_j - y_m = 0 - 10 = -10, \qquad \gamma_i = x_m - x_j = 20 - 40 = -20$$

$$\beta_j = y_m - y_i = 10 - 0 = 10, \qquad \gamma_j = x_i - x_m = 0 - 20 = -20$$

$$\beta_m = y_i - y_j = 0 - 0 = 0, \qquad \gamma_m = x_j - x_i = 40 - 0 = 40$$

$$\{f_T^{(1)}\} = \frac{(12.5\times10^{-6})(10\times10^{6})(1)(50)}{2(1-0.3)}\begin{Bmatrix}-10\\-20\\10\\-20\\0\\40\end{Bmatrix}$$

$$\{f_T^{(1)}\} = \begin{Bmatrix}-44643\\89286\\44643\\-89286\\0\\178572\end{Bmatrix}$$

Element 2

$$i = 2, j = 3, m = 5$$

$$\beta_i = 20 - 10 = 10, \qquad \gamma_i = 20 - 40 = -20$$

$$\beta_j = 10 - 0 = 10, \qquad \gamma_j = 40 - 20 = 20$$

$$\beta_m = 0 - 20 = -20, \qquad \gamma_m = 40 - 40 = 0$$

$$\{f_T^{(2)}\} = 4464.3\begin{Bmatrix}10\\-20\\10\\20\\-20\\0\end{Bmatrix} = \begin{Bmatrix}44643\\-89286\\44643\\89286\\-89286\\0\end{Bmatrix}$$

Element 3

$$i = 3, j = 4, m = 5$$

$$\beta_i = 20 - 10 = 10 \qquad \gamma_j = 20 - 0 = 20$$

$$\beta_j = 10 - 20 = -10 \qquad \gamma_j = 40 - 20 = 20$$

$$\beta_m = 20 - 20 = 0 \qquad \gamma_m = 0 - 40 = -40$$

$$\{f_T^{(3)}\} = 4464.3\begin{Bmatrix} 10 \\ 20 \\ -10 \\ 20 \\ 0 \\ -40 \end{Bmatrix} = \begin{Bmatrix} 44643 \\ 89286 \\ -44643 \\ 89286 \\ 0 \\ -178572 \end{Bmatrix}$$

Element 4

$$i = 4, j = 1, m = 5$$

$$\beta_i = 0 - 10 = -10 \qquad \gamma_i = 20 - 0 = 20$$

$$\beta_j = 10 - 20 = -10 \qquad \gamma_j = 0 - 20 = -20$$

$$\beta_m = 20 - 0 = 20 \qquad \gamma_m = 0 - 0 = 0$$

$$\{f_T^{(4)}\} = 4464.3\begin{Bmatrix} -10 \\ 20 \\ -10 \\ -20 \\ 20 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -44643 \\ 89286 \\ -44643 \\ -89286 \\ 89286 \\ 0 \end{Bmatrix}$$

$$\{F_0\} = [K]\,\{d\}$$

By direct superposition, we have

$$\begin{Bmatrix} -89,286 \\ -178,572 \\ 89,286 \\ -178,572 \\ 89,286 \\ 178,572 \\ -89,286 \\ 178,572 \\ 0 \\ 0 \end{Bmatrix} = \frac{10\times10^6}{4.16}\begin{bmatrix} 3 & 2 & 0.1 & 0.2 & 0 & 0 & -0.1 & -0.2 & -3 & -2 \\ & 6 & -0.2 & 2.6 & 0 & 0 & 0.2 & -2.6 & -2 & -6 \\ & & 3 & -2 & -0.1 & 0.2 & 0 & 0 & -3 & 2 \\ & & & 6 & -0.2 & -2.6 & 0 & 0 & 2 & -6 \\ & & & & 3 & 2 & 0.1 & 0.2 & -3 & -2 \\ & & & & & 6 & -0.2 & 2.6 & -2 & -6 \\ & & & & & & 3 & -2 & -3 & 2 \\ & & & & & & & 6 & 2 & -6 \\ & & & & & & & & 12 & 0 \\ & \text{Symmetry} & & & & & & & & 24 \end{bmatrix}$$

$$\times \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ \vdots \\ \vdots \\ u_5 \\ v_5 \end{Bmatrix}$$

Solving

$$0 = \frac{10\times10^6}{4.16}\;12\,u_s \Rightarrow u_s = 0$$

$$0 = \frac{10\times10^6}{4.16}\;24\,v_s \Rightarrow v_s = 0$$

Stresses

$$\{\sigma\} = \{\sigma_L\} - \{\sigma_T\}$$

$$\{\sigma_L\} = [D]\,[B]\,\{d\} = 0 \text{ as } \{d\} = \underline{0}$$

$$\therefore \quad \{\sigma\} = -\{\sigma_T\} = -[D]\,\{\varepsilon_T\}$$

Element 1

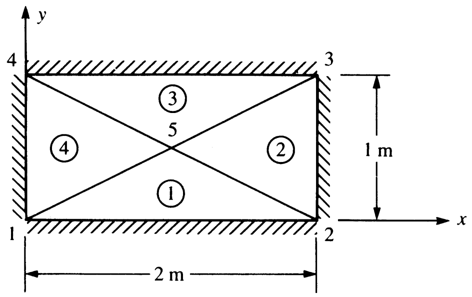
$$\{\sigma\} = -\frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}\begin{Bmatrix} \alpha T \\ \alpha T \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \tau_{xy} \end{Bmatrix} = \frac{-10\times10^6}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{Bmatrix} 6.25\times10^{-4} \\ 6.25\times10^{-4} \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -8929 \\ -8929 \\ 0 \end{Bmatrix}\text{psi}$$

Since $[D]$ and $\{\varepsilon_T\}$ are same for all elements, all element stresses are equal.

15.18



Based on use of symmetry

$$u_s = v_s = 0 \text{ (Also see solution to Problem 15.17)}$$

$$\therefore \quad \{\sigma\} = \{\sigma_L\} - \{\sigma_T\} = [D]\,[B]\,\{d\} - [D]\,\{\varepsilon_T\}$$

$$\{\sigma\} = -[D]\,\{\varepsilon_T\}$$

All stresses in elements are equal

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{-E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}\begin{Bmatrix} \alpha T \\ \alpha T \\ 0 \end{Bmatrix}$$

$$= \frac{-210\times10^9}{1-0.25^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{0} \end{bmatrix}\begin{Bmatrix} 12\times10^{-6}(-20) \\ 12\times10^{-6}(-20) \\ 0 \end{Bmatrix}$$