$\alpha_m = r_i z_j - z_i r_j = 0,\ \beta_m = z_i - z_j = 0 - 0 = 0$

$\gamma_m = r_j - r_i = b - 0 = b$

So the shape functions evaluated at $r = b$ and $z = z$

$$N_i = \frac{1}{bh}\ (bh + (-h)\, b + 0\ \ z) = 0$$

$$N_j = \frac{1}{bh}\ (0 + h\, b + (-b)z) = \frac{1}{bh}\ (hb - bz)$$

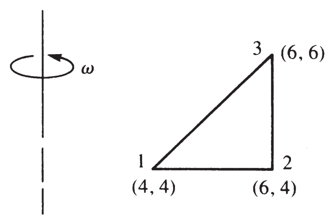
$$N_m = \frac{1}{bh}\ (0 + 0\ b + b\ z) = \frac{1}{bh}\ (bz)$$

$$\{f_s\} = \int_0^h \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{bh}(hb-bz) & 0 \\ 0 & \frac{1}{bh}(hb-bz) \\ \frac{1}{bh}(bz) & 0 \\ 0 & \frac{1}{bh}(bz) \end{bmatrix} \begin{Bmatrix} \frac{p_0\, z}{h} \\ 0 \end{Bmatrix} 2\pi\ dz$$

$$= \frac{2\pi b}{bh}\int_0^h \begin{Bmatrix} 0 \\ 0 \\ p_0 bz - \frac{p_0 bz^2}{h} \\ 0 \\ \frac{p_0 bz^2}{h} \\ 0 \end{Bmatrix} dz = \frac{2\pi}{h}\begin{Bmatrix} 0 \\ 0 \\ \frac{p_0 bz^2}{2} - \frac{p_0 bz^3}{3h} \\ 0 \\ \frac{p_0 bz^3}{3h} \\ 0 \end{Bmatrix}\Bigg|$$

$$= \frac{2\pi}{h}\begin{Bmatrix} 0 \\ 0 \\ \frac{p_0 bh^2}{6} \\ 0 \\ \frac{p_0 bh^2}{3} \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{s1x} \\ f_{s1y} \\ f_{s2x} \\ f_{s2y} \\ f_{s3x} \\ f_{s3y} \end{Bmatrix} = 2\pi b \begin{Bmatrix} 0 \\ 0 \\ \frac{p_0\, h}{6} \\ 0 \\ \frac{p_0\, h}{3} \\ 0 \end{Bmatrix}$$

9.3



Equation to be evaluated is $\{f_B\} = \dfrac{2\pi\bar{r}A}{3}\begin{Bmatrix}\bar{R}_B\\ Z_B\\ \bar{R}_B\\ Z_B\\ \bar{R}_B\\ Z_B\end{Bmatrix}$

$$\bar{r} = 4 + 2 \times \frac{2}{3} \quad \Rightarrow \bar{r} = 5.333 \text{ in.}$$

$$Z_B = 0.283 \ \frac{\text{lb}}{\text{in.}^3}$$

$$\bar{R}_B = w^2 \rho \ \bar{r} = \left[20\,\frac{\text{rev.}}{\text{min}}\left(2\pi\frac{\text{rad}}{\text{rev}}\right)\frac{1 \text{ min}}{60\,s}\right]^2 \frac{\left(0.283\,\frac{\text{lb}}{\text{in.}^3}\right)}{\left(32.2\times 12\,\frac{\text{lb}}{\text{in.}^3}\right)} \ [5.333 \text{ m}]$$

$$\bar{R}_B = 0.01712 \ \frac{\text{lb}}{\text{in.}^3}$$

$$\frac{2\pi\bar{r}A}{3} = \frac{2}{3}\pi\,(5.333 \text{ in.})\ (2 \text{ in.}^2) = 22.34 \text{ in.}^3$$

So

$$f_{B1r} = (22.34)\ (0.01712) = 0.382 \text{ lb}$$

$$f_{B1z} = (-22.34)\ (0.283) = -\ 6.32 \text{ lb}$$

$$f_{B2r} = (22.34)\ (0.01712) = 0.382 \text{ lb}$$

$$f_{B3z} = (-\ 22.34)\ (0.283) = -\ 6.32 \text{ lb}$$

$$f_{B3r} = (22.34)\ (0.01712) = 0.382 \text{ lb}$$

$$f_{B3z} = (-\ 22.34)\ (0.283) = -\ 6.32 \text{ lb}$$

9.4

(a) Element Figure 9.4 a

The equation to be evaluated is $\{\sigma\} = [D]\ [B]\ \{d\}$

$r_i = 0,\ z_i = 0,\ r_j = 2,\ z_j = 0,\ r_m = 1,\ z_m = 3$

$\alpha_i = 6,\ \alpha_j = 0,\ \alpha_m = 0,\ \beta_i = -\ 3,\ \beta_j = 3,\ \beta_m = -\ 3,$

$\gamma_i = -\ 1,\ \gamma_j = -\ 1,\ \gamma_m = 2$

$\bar{r} = 1,\ \bar{z} = 1,\ A = \dfrac{1}{2}\ (3)\ (2) = 3 \text{ in.}^2$

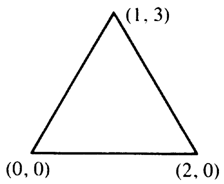


Figure 9.4a

$$[\bar{B}] = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 & 0 & -3 & 0\\ 0 & -1 & 0 & -1 & 0 & 2\\ 2 & 0 & 2 & 0 & -1 & 0\\ -1 & -3 & -1 & 3 & 2 & -3 \end{bmatrix}$$

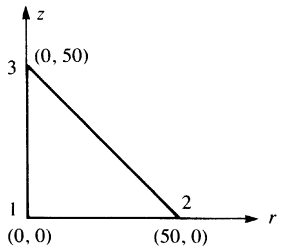
$$[D] = \frac{30\times 10^6}{(1+0.25)(1-0.5)}\begin{bmatrix} 0.75 & 0.25 & 0.25 & 0\\ 0.25 & 0.75 & 0.25 & 0\\ 0.25 & 0.25 & 0.75 & 0\\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{pmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

$$[k] = 2\pi r A\, [B]^T [D]\, [B]$$

$$[k] = \begin{pmatrix} 8.577\times10^8 & 1.759\times10^8 & -3.738\times10^8 & -8.796\times10^7 & 4.398\times10^7 & -8.796\times10^7 \\ 1.759\times10^8 & 4.618\times10^8 & -8.796\times10^7 & -6.597\times10^7 & -3.519\times10^8 & -3.958\times10^8 \\ -3.738\times10^8 & -8.796\times10^7 & 1.561\times10^9 & -3.519\times10^8 & 3.958\times10^8 & 4.398\times10^8 \\ -8.796\times10^7 & -6.597\times10^7 & -3.519\times10^8 & 4.618\times10^8 & 1.759\times10^8 & -3.958\times10^8 \\ 4.398\times10^7 & -3.519\times10^8 & 3.958\times10^8 & 1.759\times10^8 & 6.158\times10^8 & 1.759\times10^8 \\ -8.796\times10^7 & -3.958\times10^8 & 4.398\times10^8 & -3.958\times10^8 & 1.759\times10^8 & 7.917\times10^8 \end{pmatrix}$$

9.7 (a)



(a)

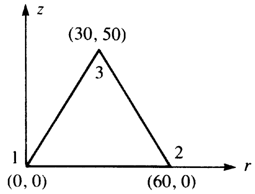
From Problem 9.6 (a), we have $[D]\ [\bar{B}]$

$$\therefore\quad \{\sigma\} = [D]\ [\bar{B}]\{d\}$$

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{210\times10^3\,\text{MPa}}{(1250)(1.25)}\begin{bmatrix} -25 & -12.5 & 50 & 0 & 12.5 & 12.5 \\ 0 & -37.5 & 25 & 0 & 12.5 & 37.5 \\ 25 & -12.5 & 50 & 0 & 37.5 & 12.5 \\ -12.5 & -12.5 & 0 & 12.5 & 12.5 & 0 \end{bmatrix}\begin{Bmatrix} 0.05 \\ 0.03 \\ 0.02 \\ 0.02 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -84 \\ -84 \\ 252 \\ -101 \end{Bmatrix}\ \text{MPa}$$

(b)



(b)

From Problem 9.6 (b), we have $[D]\ [\bar{B}]$

$$\therefore \begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{210 \times 10^3}{(1500)(1.25)} \begin{bmatrix} -37.5 & 0 & 52.5 & -15 & 7.5 & 15 \\ -7.5 & 0 & 22.5 & -45 & 7.5 & 45 \\ 7.5 & 0 & 37.5 & -15 & 22.5 & 15 \\ 0 & -15 & -15 & 15 & 15 & 0 \end{bmatrix} \begin{Bmatrix} 0.05 \\ 0.03 \\ 0.02 \\ 0.02 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -103 \\ -103 \\ 112 \\ -73 \end{Bmatrix} \text{MPa}$$

(c) $u_i = 0.00005 \qquad w_i = 0.00003$

$u_j = 0.00002 \qquad w_j = 0.00002$

$u_m = 0 \qquad w_m = 0$

$$\{d\} = \begin{pmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{pmatrix}$$

$$\begin{pmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{pmatrix} = [D]\,[B]\,\{d\}$$

$\sigma_r = -\,2.87 \times 10^9$ Pa

$\sigma_z = -\,2.45 \times 10^9$ Pa

$\sigma_\theta = 3.57 \times 10^9$ Pa

$\tau_{rz} = -\,1.89 \times 10^9$ Pa

9.8 No, not in general, as the axisymmetric elements are rings, not plane quadrilaterals or triangles. So axisymmetric nodes are actually nodal circles whereas plane stress elements have node points.

9.9 No, the element circumferential strain is a function of r and z (see Equation (9.1.15).

9.10 Make $u_r = 0$ for all nodes acting on the axis of symmetry.

9.11 How would you evaluate circumferential strain ε_θ at $r = 0$?

From text Equation (9.1.15)

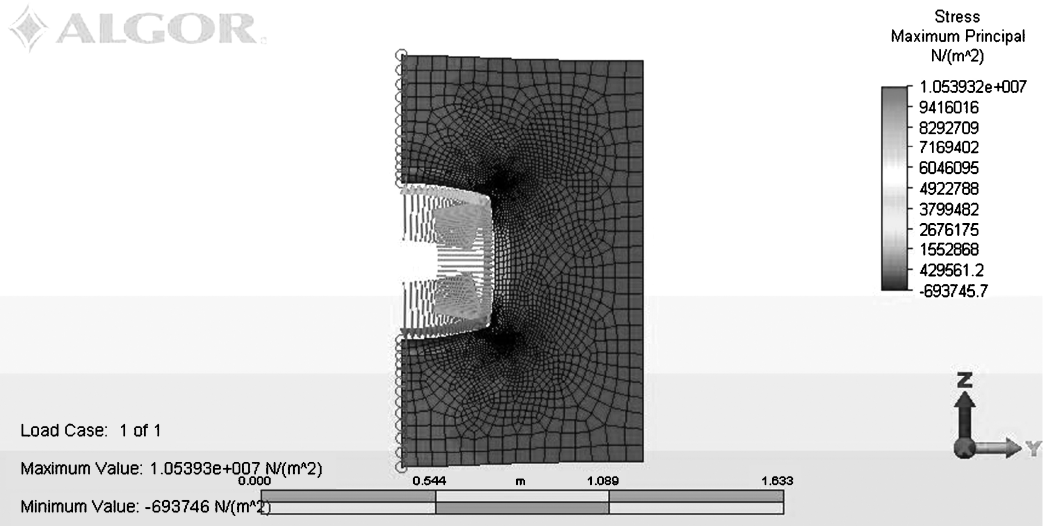
$$\varepsilon_\theta = \frac{a_1}{r} + a_2 + \frac{a_3 z}{r} \tag{1}$$

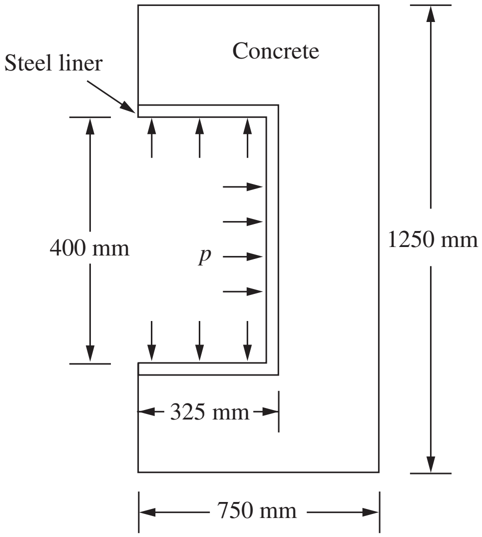
$$\varepsilon_r = a_2 \tag{2}$$

Also from text Equation (9.1.1e)

$$\varepsilon_\theta = \frac{u}{r} \tag{3}$$

9.15





Note: Without the arc (inside radius), we have a 90° re-entrant corner where stress is approaching infinity. We have a singularity in the linear-elastic solution based on linear theory of elasticity. Therefore, we need the arc as in good practice or elastic-plastic model where an upper bound on the corner stress is the yield strength of the material.