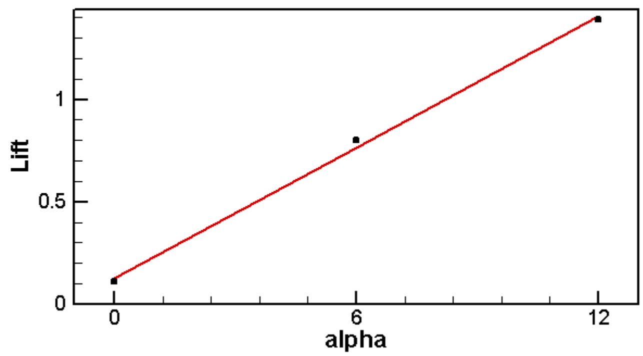
ME-400: Solution to Exam 2 (Fall 2013)

1. a)



The linear least squares approximation $C_L(\alpha) = a + b\alpha$ is obtained from

$$n\,a + \left(\sum_{i=1}^{3}\alpha_i\right)b = \sum_{i=1}^{3}(C_L)_i$$
$$\left(\sum_{i=1}^{3}\alpha_i\right)a + \left(\sum_{i=1}^{3}\alpha_i^2\right)b = \sum_{i=1}^{3}(C_L)_i\,\alpha_i$$

$$n = 3$$

$$\left(\sum_{i=1}^{3}\alpha_i\right) = 0 + 6 + 12 = 18 \qquad \sum_{i=1}^{3}(C_L)_i = 0.11 + 0.8 + 1.39 = 2.3$$

$$\left(\sum_{i=1}^{3}\alpha_i^2\right) = 0 + 36 + 144 = 180 \qquad \sum_{i=1}^{3}(C_L)_i\,\alpha_i = 0.11\times 0 + 0.8\times 6 + 1.39\times 12 = 21.48$$

$$\left.\begin{array}{r} 3a + 18b = 2.3 \\ 18a + 180b = 21.48 \end{array}\right\} \Rightarrow C_L(\alpha) = 0.12667 + 0.10667\alpha$$

b)

$$L_2^0(\alpha) = \frac{(\alpha-6)(\alpha-12)}{(0-6)(0-12)} = 0.013889\alpha^2 - 0.25\alpha + 1$$

$$L_2^1(\alpha) = \frac{(\alpha-0)(\alpha-12)}{(6-0)(6-12)} = -0.027778\alpha^2 + 0.33333\alpha$$

$$L_2^2(\alpha) = \frac{(\alpha-0)(\alpha-6)}{(12-0)(12-6)} = 0.013889\alpha^2 - 0.083333\alpha$$

$$p_2(\alpha)=L_2^0(\alpha)\times 0.11+L_2^1(\alpha)\times 0.8+L_2^2(\alpha)\times 1.39=-0.0013889\alpha^2+0.12333\alpha+0.11$$

c)

$$L_1^0(\alpha)=\frac{(\alpha-12)}{(0-12)}=-0.083333\alpha+1\ ,\ L_1^1(\alpha)=\frac{(\alpha-0)}{(12-0)}=0.083333\alpha$$

$$p_1(\alpha)=L_1^0(\alpha)\times 0.11+L_1^1(\alpha)\times 1.39=0.10667\alpha+0.11$$

d) The error is $\left|p_2(\alpha)-p_1(\alpha)\right|=\left|\frac{\alpha(\alpha-12)}{2!}f^{(2)}(c)\right|\le\left|\frac{12^2}{8}f^{(2)}(c)\right|$

$f''(x)=2\times 0.0013889=0.0027778$, hence

$$\left|p_2(\alpha)-p_1(\alpha)\right|=\le\left|\frac{12^2}{8}f^{(2)}(c)\right|\le\frac{144}{8}\times 0.0027778=0.0500004$$

2. a) The error for the Trapezoidal rule is $E_n^T(f)=-\frac{h^2(b-a)}{12}f''(c)$, where $f''(x)=(2+x)e^x$. Set $h=\frac{b-a}{n}$

Then we have $E_n^T(f)=-\frac{h^2}{12}f''(c)=-\frac{1}{12n^2}(2+c)e^c$, and we want

$\left|E_n^S(f)\right|=\left|-\frac{1}{12n^4}(2+c)e^c\right|\le 5\times 10^{-6}$. Taking $\max\limits_{1\le c\le 2}(2+c)e^c=4e^2\doteq 29.556$ we get

$$\left|E_n^S(f)\right|=\left|-\frac{1}{12n^2}(2+c)e^c\right|\le\frac{1}{12n^2}29.556\le 5\times 10^{-6}\ \Rightarrow\ n>701.8\ \Rightarrow\ n=702$$

b) $T_1\left(f(x)\right)=\frac{h}{2}\left(f(1)+f(2)\right)=\frac{1}{2}\left(e+2e^2\right)\doteq 8.748197$

$$T_2\left(f(x)\right)=\frac{h}{2}\left(f(1)+2f(1.5)+f(2)\right)\doteq 7.735365$$

$$R_2\left(f(x)\right)=\frac{1}{2^2-1}\left[2^2T_2-T_1\right]\doteq 7.397755$$

c) $E_2^T(f)\approx-\frac{h^2}{12}[f'(b)-f'(a)]$, $a=1,\ b=2,\ h=1/2$, $f'(x)=(1+x)e^x$

$$E_2^T(f)\approx-\frac{1}{4\times 12}\left[3e^2-2e\right]=-0.34856$$

d) $I\left(f(x)\right)=\int_{1}^{2} xe^{x}dx=\int_{-1}^{1}\left(\frac{3+t}{2}\right)e^{\left(\frac{3+t}{2}\right)}\frac{1}{2}dt=\int_{-1}^{1}g(t)dt$

$$I_2\left(g(t)\right)=w_1 g(t_1)+w_2 g(t_2)$$
$$w_1=1.0 \qquad t_1=-0.577350269$$
$$w_2=1.0 \qquad t_2=0.577350269$$

$$I_3(g(x))=1.0\times 2.033773+1.0\times 5.3495$$

$$I_3(g(x))=7.383273$$

Compared to the exact solution the 2 point Gauss quadrature gives slightly better accuracy than the Richardson extrapolation on the Trapezoidal rule with less work.