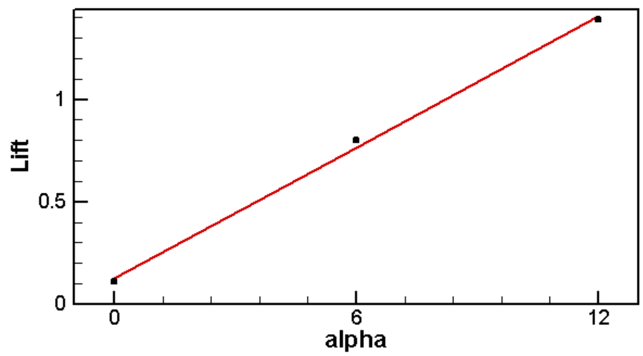
ME-500: Solution to Exam 2 (Fall 2013)

1. a)



$$C_L(\alpha) = a + b\alpha$$

$$n\,a + \left(\sum_{i=1}^{3}\alpha_i\right)b = \sum_{i=1}^{3}(C_L)_i$$

$$\left(\sum_{i=1}^{3}\alpha_i\right)a + \left(\sum_{i=1}^{3}\alpha_i^2\right)b = \sum_{i=1}^{3}(C_L)_i\,\alpha_i$$

$$n = 3$$

$$\left(\sum_{i=1}^{3}\alpha_i\right) = 0 + 6 + 12 = 18 \qquad \sum_{i=1}^{3}(C_L)_i = 0.11 + 0.8 + 1.39 = 2.3$$

$$\left(\sum_{i=1}^{3}\alpha_i^2\right) = 0 + 36 + 144 = 180 \qquad \sum_{i=1}^{3}(C_L)_i\,\alpha_i = 0.11\times 0 + 0.8\times 6 + 1.39\times 12 = 21.48$$

$$\left.\begin{aligned} 3a + 18b &= 2.3 \\ 18a + 180b &= 21.48 \end{aligned}\right\} \Rightarrow C_L(\alpha) = 0.12667 + 0.10667\alpha$$

b)

$$L_2^0(\alpha) = \frac{(\alpha-6)(\alpha-12)}{(0-6)(0-12)} = 0.013889\alpha^2 - 0.25\alpha + 1$$

$$L_2^1(\alpha) = \frac{(\alpha-0)(\alpha-12)}{(6-0)(6-12)} = -0.027778\alpha^2 + 0.33333\alpha$$

$$L_2^2(\alpha) = \frac{(\alpha-0)(\alpha-6)}{(12-0)(12-6)} = 0.013889\alpha^2 - 0.083333\alpha$$

$$p_2(\alpha) = L_2^0(\alpha)\times 0.11 + L_2^1(\alpha)\times 0.8 + L_2^2(\alpha)\times 1.39 = -0.0013889\alpha^2 + 0.12333\alpha + 0.11$$

Define $D = p_2(\alpha) - C_L(\alpha) = -0.0013889\alpha^2 + 0.12333\alpha + 0.11 - (0.12667 + 0.10667\alpha)$

$$|D| = \left|-0.0013889\alpha^2 + 0.016666\alpha - 0.01667\right|$$

$$\frac{\partial D}{\partial \alpha} = -0.0027778\alpha + 0.01667 = 0 \;\Rightarrow\; \alpha = 6.00115 \;\Rightarrow\; \max|D| = 0.0333$$

c) The error is $\left|f(\alpha) - p_2(\alpha)\right| = \left|\frac{\alpha(\alpha-6)(\alpha-12)}{3!} f^{(3)}(c)\right| \le \left|\frac{6^3}{9\sqrt{3}} f^{(3)}(c)\right|$

$$f^{(3)}(c) \cong f[0,6,12,18]\times 3! = 6\times\frac{f[6,12,18] - f[0,6,12]}{18} = \frac{\frac{f[12,18]-f[6,12]}{12} - \frac{f[6,12]-f[0,6]}{12}}{3}$$

$$f^{(3)}(c) \cong \frac{f[12,18] - 2f[6,12] + f[0,6]}{36} = \frac{\frac{f(18)-f(12)}{6} - \frac{f(12)-f(6)}{3} + \frac{f(6)-f(0)}{6}}{36}$$

$$f^{(3)}(c) \cong \frac{f(18) - 3f(12) + 3f(6) - f(0)}{216} = \frac{1.55 - 3\times(1.39-0.8) + 0.11}{216} \doteq -0.0005093$$

So an error estimate is $\left|f(\alpha) - p_2(\alpha)\right| \le \left|\frac{6^3}{9\sqrt{3}} f^{(3)}(c)\right| \cong \frac{6^3}{9\sqrt{3}} 0.0005093 \doteq 0.007056$

2. If α is a root, a sufficient condition for convergence is $|g'(\alpha)| < 1$.

For method 1, $g_1'(x) = \frac{3x^2 + 10x}{17} \;\Rightarrow\; g_1'(-7) = \frac{3\times 49 - 10\times 7}{17} \doteq 4.53$ hence it diverges.

For method 2,

$$g_2'(x) = \frac{(17-10x)\times x^2 - 2x\times(21 + 17x - 5x^2)}{x^4} = \frac{-42 - 17x}{x^3} \;\Rightarrow\; g_2'(-7) = \frac{-42 + 17\times 7}{-343} \doteq -0.2245$$

Hence method 2 converges.

3. a) The error for Simpson's rule is $E_n^S(f) = -\frac{h^4(b-a)}{180} f^{(4)}(c)$, where $f^{(4)}(x) = (4+x)e^x$. Set $h = \frac{b-a}{n}$

Then we have $E_n^S(f) = -\frac{h^4}{180} f^{(4)}(c) = -\frac{1}{180n^4}(4+c)e^c$, and we want

$\left|E_n^S(f)\right| = \left|-\frac{1}{180n^4}(4+c)e^c\right| \le 5\times10^{-6}$. Taking $\max_{1\le c\le 2}(4+c)e^c = 6e^2 \doteq 44.334$ we get

$\left|E_n^S(f)\right| = \left|-\frac{1}{180n^4}(4+c)e^c\right| \le \frac{1}{180n^4}44.334 \le 5\times10^{-6} \;\Rightarrow\; n > 14.9 \;\Rightarrow\; n = 15$

b) $S_1(f(x)) = \frac{h}{3}\left(f(1) + 4f(1.5) + f(2)\right) = \frac{1}{6}\left(e + 6e^{1.5} + 2e^2\right) \doteq 7.397755$

$S_2(f(x)) = \frac{h}{3}\left(f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)\right) \doteq 7.389616$

$R_2(f(x)) = \frac{1}{2^4-1}\left[2^4 S_2 - S_1\right] \doteq 7.389073$

c) $I(f(x)) = \int_1^2 xe^x dx = \int_{-1}^{1}\left(\frac{3+t}{2}\right)e^{\left(\frac{3+t}{2}\right)}\frac{1}{2}dt = \int_{-1}^{1} g(t)dt$

$I_3(g(t)) = w_1 g(t_1) + w_2 g(t_2) + w_3 g(t_3)$

$w_1 = 0.555555556 \quad t_1 = -0.774596669$

$w_2 = 0.888888889 \quad t_2 = 0.0$

$w_3 = 0.555555556 \quad t_3 = 0.774596669$

$I_3(g(x)) = 0.555555556\times1.69273481 + 0.888888889\times3.3612668 + 0.555555556\times6.22950894$

$I_3(g(x)) = 7.389039$

Compared to the exact solution the 3 point Gauss quadrature gives slightly better accuracy than the Richardson extrapolation on Simpson's rule with much less work.