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HomeWork#5

3.4

Problem 1 (a) Calculate the first six iterates in the iteration

$$x_{n+1} = 1 + 0.3 \sin x_n$$

with $x_0 = 1$. Choose other initial guesses x_0 and repeat this calculation.
(b) Find an interval $[a, b]$ satisfying the hypotheses of Theorem 3.4.2.
Hint: For $g(x) = 1 + 0.3 \sin x$, let

$$a = \min_{-\infty<x<\infty} g(x), \qquad b = \max_{-\infty<x<\infty} g(x)$$

(c) Prepare a table in the same manner as Table 3.5 in the text. The true solution is $\alpha = 1.28809131321184$.

Solution: (a)

n	$x_0 = 1$	$x_0 = 2$
1	1.252441295	1.272789228
2	1.284925476	1.286777060
3	1.287824933	1.287981080
4	1.288069011	1.288082087
5	1.288089447	1.288090541
6	1.288091157	1.288091249

(b) Since $-1 \le \sin x \le 1$,

$$a = \min_{-\infty<x<\infty} g(x) = 1 + 0.3 \cdot (\min_{-\infty<x<\infty} \sin x) = 1 + 0.3 \cdot (-1) = 0.7$$
$$b = \max_{-\infty<x<\infty} g(x) = 1 + 0.3 \cdot (\max_{-\infty<x<\infty} \sin x) = 1 + 0.3 \cdot (1) = 1.3$$

Thus, the interval $[0.7, 1.3]$ satisfies the hypotheses of Theorem 3.4.2. This also confirms the results in (a).

(c) Use the same initial guess $x_0 = 1$ as (a).

n	x_n	$\alpha - x_n$	r_n
0	1.0	2.88E−1	
1	1.252441295	3.56E−2	0.1237
2	1.284925476	3.17E−3	0.0888
3	1.287824933	2.66E−4	0.0841
4	1.288069011	2.23E−5	0.0837
5	1.288089447	1.87E−6	0.0837
6	1.288091157	1.56E−7	0.0837
7	1.288091300	1.32E−8	0.0845

$g'(\alpha) = 0.3 \cos \alpha \doteq 0.083686$ which shows the values of r_n converge to $g'(\alpha)$.

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Problem 3 How many solutions are there to the equation $x = e^{-x}$? Will the iteration $x_{n+1} = e^{-x_n}$ converge for suitable choices of x_0? Calculate the first six iterates when $x_0 = 0$.

Solution: The equation $x = e^{-x}$ has only one solution. x and e^{-x} are monotone increasing and monotone decreasing, respectively. In addition, $x > e^{-x}$ as $x > 1$ and $x < e^{-x}$ as $x < 0$. Therefore, this equation has exactly one solution.

The iteration $x_{n+1} = e^{-x_n}$ will converge for suitable choices of x_0. For example, choose $a = 0.1$ and $b = 1$. Then $0.1 \le x \le 1$ implies $0.1 \le e^{-x} \le 1$. Compute the derivative of e^{-x},

$$| (e^{-x})' | = | -e^{-x} | = e^{-x} < 1 \qquad \text{if } x > 0$$

Therefore, the iteration is convergent when $x_0 \in [a, b]$.

The true solution is $\alpha \doteq 0.56714329040978$. We give the first six iterates when $x_0 = 0$ as follows:

n	x_n
0	0.0
1	1.0
2	0.3678794412
3	0.6922006276
4	0.5004735006
5	0.6062435351
6	0.5453957860

Problem 7 What are the solutions α, if any, of the equation $x = \sqrt{1+x}$? Does the iteration $x_{n+1} = \sqrt{1+x_n}$ converge to any of these solutions (assuming x_0 is chosen sufficiently close to α)?

Solution: The equation has one solution. Squaring both sides of the equation, we have

$$x^2 - x - 1 = 0.$$

Then the two roots are

$$\frac{1+\sqrt{5}}{2}, \qquad \frac{1-\sqrt{5}}{2}$$

The second root is negative; and since the original equation has only one root, and it is positive, we have

$$\alpha = \frac{1+\sqrt{5}}{2}$$

The iteration $x_{n+1} = \sqrt{1+x_n}$ converges to α when $x_0 \in [1, 2]$. Note that $1 \le x \le 2$ implies $1 \le \sqrt{1+x} \le 2$, and

$$g'(x) = \frac{1}{2\sqrt{1+x}} < \frac{1}{2}, \qquad x \in [1, 2]$$

Problem 8 Which of the following iterations will converge to the indicated α, provided x_0 is chosen sufficiently close to α? If it does converge, determine the convergence order.

(a) $x_{n+1} = \frac{15x_n^2 - 24x_n + 3}{4x_n}, \quad \alpha = 1.$

(b) $x_{n+1} = \frac{3}{4}x_n + 1/x_n^3, \quad \alpha = \sqrt{2}.$

Solution: We apply Theorem 3.4.2.

(a)

$$g'(1) = \frac{24(5x-4)(15x^2-24x+13)}{16x^2} = 3 > 1,$$

So the iteration method does not converge

(b) $g'(\sqrt{2}) = 0$ and $g''(\sqrt{2}) \neq 0$. So the method converges for sufficiently good initial guess and the order of the convergence is 2.

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Problem 11 The iteration $x_{n+1} = 2 - (1+c)x_n + cx_n^3$ will converge to $\alpha = 1$ for some values of c (provided that x_0 is chosen sufficiently close to α). Find the values of c for which convergence occurs. For what values of c, if any, will the convergence be quadratic?

Solution: Let $g(x) = 2 - (1+c)x + cx^3$. The iteration converges if $|g'(x)| < 1$ for x sufficiently close to α.

$$|g'(x)| = |-(1+c) + 3cx^2| < 1$$

Replace x by $\alpha = 1$,

$$|-(1+c) + 3c| < 1 \qquad \Longrightarrow \qquad 0 < c < 1$$

The convergence is quadratic if $g'(\alpha) = 0$ and $g''(\alpha) \neq 0$.

$$0 = g'(\alpha) = -(1+c) + 3c = 2c - 1$$

$$\Longrightarrow \quad c = \frac{1}{2}$$

$$g''(\alpha) = 6c(1) = 6\frac{1}{2} \neq 0$$

This shows the convergence is quadratic if $c = 1/2$.

Problem 14 For slowly convergent sequences, the Aitken extrapolation formula (3.52) can greatly accelerate the convergence. Use the following algorithm:

$$\begin{aligned}
x_1 &= g(x_0) \\
x_2 &= g(x_1) \\
x_3 &= \text{Aitken extrapolate of } x_0,\ x_1, \text{ and } x_2. \\
x_4 &= g(x_3) \\
x_5 &= g(x_4) \\
x_6 &= \text{Aitken extrapolate of } x_3,\ x_4, \text{ and } x_5.
\end{aligned}$$

Continue this process in the same manner. Apply it to the following iterations.
(a) $x_{n+1} = 2e^{-x_n}$, $x_0 = 0.8$
(b) $x_{n+1} = 0.9/(1 + x_n^4)$, $x_0 = 0.75$
(c) $x_{n+1} = 6.28 + \sin x_n$, $x_0 = 6$

Solution: We give the first iterates for each iteration.

n	(a)	(b)	(c)
1	0.8986579282	0.6836795252	6.000584502
2	0.8142313434	0.7386255967	6.001145771
3	0.8531633541	0.7137294414	6.014705057
4	0.8521300068	0.7145702618	6.014733561
5	0.8530110082	0.7138759467	6.014761044
6	0.8530261532	0.7138620683	6.014761095

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Problem 18 What is the order of convergence of the iteration

$$x_n = \frac{x_n(x_n^2+3a)}{3x_n^2+a}$$

as it converges to the fixed point $\alpha = \sqrt{a}$?

Solution: Compute $g'(x)$, $g''(x)$, and $g^{(3)}(x)$.

$$g'(x) = \frac{3(x^2+a)}{3x^2+a} - \frac{6x^2(x^2+3a)}{(3x^2+a)^2} \qquad g'(\sqrt{a}) = 0$$

$$g''(x) = \frac{48ax(x^2-a)}{(3x^2+a)^3} \qquad g''(\sqrt{a}) = 0$$

$$g^{(3)}(x) = \frac{-48a(9x^4-18ax^2+a^2)}{(3x^2+a)^4} \qquad g^{(3)}(\sqrt{a}) = \frac{1.5}{a}$$

These show the iteration is of order 3.

3.5

Problem 1 Use Newton's method to calculate the roots of

$$f(x) = x^5 + 0.9x^4 - 1.62x^3 - 1.458x^2 + 0.6561x + 0.59049$$

Print out the iterates and the function values. Produce the ratios of (3.62) by using the approximation (3.65),

$$\frac{\alpha - x_n}{\alpha - x_{n-1}} \doteq \frac{x_{n+1}-x_n}{x_n - x_{n-1}}$$

Repeat the problem for several choices of x_0. Make observations that seem important relative to the rootfinding problem.
Note: The above will first approach $\lambda = (m-1)/m$, as in (3.61), but they will then depart from it due to noise in evaluation of $f(x)$ as x_n approaches the root.

Solution: Graphically, there are roots near -1 and 1. We used subroutine NEWTON with $\epsilon = 10^{-8}$.
(a)

initial guess $x_0 = -1$

n	x_n	RATIO	n	x_n	RATIO
1	−0.9677970		7	−0.9063234	0.6364762
2	−0.9457323	0.6851745	8	−0.9044989	0.6270304
3	−0.9307379	0.6795648	9	−0.9032919	0.6615269
4	−0.9206112	0.6753669	10	−0.9021632	0.9351111
5	−0.9138051	0.6720974	11	−0.9021632	0.0000000
6	−0.9092333	0.6717168			

The program converged in 11 iterates to $\alpha \approx -0.9021632$. The ratios stayed around 2/3 for most of the iterates. Thus the multiplicity of the root appears to be $m = 3$, based on (3.61) and (3.63). With this, we look at the root of

$$f''(x) = 20x^3 + 10.8x^2 - 9.72x - 2.916$$

With the above approximate α as an initial guess, the iterates quickly converge to $\alpha = -0.9$. See the table below.

initial guess $x_0 = -.9021632$

n	x_n	RATIO
1	−0.9000103	
2	−0.9000000	0.0047897
3	−0.9000000	0.0000000

(b)

initial guess $x_0 = 1$

n	x_n	RATIO	n	x_n	RATIO
1	0.9536586		11	0.9000874	0.3268482
2	0.9279458	0.5548554	12	0.9000290	1.4583334
3	0.9142861	0.5312398	13	0.9002051	−3.0153062
4	0.9072263	0.5168368	14	0.9001305	−0.4240271
5	0.9036348	0.5087214	15	0.9000913	0.5243416
6	0.9018209	0.5050701	16	0.8999794	2.8584476
7	0.9009129	0.5005422	17	0.9002275	−2.2167199
8	0.9004606	0.4981947	18	0.9001377	−0.3619986
9	0.9002500	0.4655422	19	0.9000635	0.8261446
10	0.9001275	0.5819417	20	0.9000635	0.0000000

The program converged in 20 iterates to $\alpha \approx 0.9000635$. The ratios stayed around 1/2 for most of the iterates. Thus the multiplicity of the root appears to be $m = 2$, based on (3.61) and (3.63). With this, we look at the root of

$$f'(x) = 5x^4 + 3.6x^3 - 4.86x^2 - 2.916x + 0.6561$$

With the above approximate α as an initial guess, the iterates quickly converge to $\alpha = 0.9$. See the table below.

initial guess $x_0 = 0.9000635$

n	x_n	RATIO
1	0.9000001	
2	0.9000000	0.0009398
3	0.9000000	0.0000000

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Problem 6 Consider the problem of solving $\frac{x}{1+x} - 0.99 = 0$, calling its root α. Then let $\alpha(\epsilon)$ be the solution of $\frac{x}{1+x} - 0.99 + \epsilon = 0$.
(a) Using (3.74), estimate $\alpha(\epsilon) - \alpha$.
(b) Calculate $\alpha(\epsilon)$ directly, compute $\alpha(\epsilon) - \alpha$, and compare with (a). Comment on your results.

Solution: (a) Note the root $\alpha = 99$ and $g(x) = 1$. Then

$$\alpha(\epsilon) - \alpha \doteq -\epsilon\frac{g(\alpha)}{f'(\alpha)} = \frac{-\epsilon}{\frac{1}{(1+\alpha)^2}} = \frac{-\epsilon}{10^{-4}} = -\epsilon \times 10^4 \equiv \text{EST}$$

(b) We have computed three different $\alpha(\epsilon)$'s, with $\epsilon = -0.009, 0.09, -0.01$. EST stands for the estimation of $\alpha(\epsilon) - \alpha$ by using the formula in (a).

ϵ	$\alpha(\epsilon)$	$\alpha(\epsilon) - \alpha$	EST
−0.009	999	900	90
0.001	89.90909091	−9.09090909	−10
−0.001	110.1111111	11.1111111	10
0.09	9	−90	−900
−0.01	No root		100

When $\epsilon = -0.01$, the equation does not have solution, but the formula in (a) can not predict the vanishing of the solution. ϵ is smaller, EST gives better estimation.

Problem 8 Newton's method is used to find a root of $f(x) = 0$. The first few iterates are shown in the following table, giving a very slow speed of convergence. What can be said about the root α to explain this convergence. Knowing $f(x)$, how would you find an accurate value for α?

n	x_n	$x_n - x_{n-1}$
0	0.75	
1	0.752710	0.00271
2	0.754795	0.00208
3	0.756368	0.00157
4	0.757552	0.00118
5	0.758441	0.000889

Solution: Let

$$\text{RATIO} = \frac{x_{n+1} - x_n}{x_n - x_{n-1}}$$

Then use the table in this problem, we have

n	RATIO
1	0.7675
2	0.7548
3	0.7516
4	0.7534

Since $\lambda = (m-1)/m \doteq 0.75$, we expect a root of multiplicity 4. To find it accurately, use Newton method to solve $f^{(3)}(x) = 0$ and $x_5 = 0.758441$ as the initial guess.