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HW#6

4.1

Problem 8 Using (4.6), find the polynomial $P(x)$ that interpolates the following data. In each case, simplify (4.6) as much as possible.
(a) $\{(0, 1), (1, 2), (2, 3)\}$
Comment on your results.

Solution: (a)

$$P_2(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)}(1) + \frac{(x-0)(x-2)}{(1-0)(1-2)}(2) + \frac{(x-0)(x-1)}{(2-0)(2-1)}(3) = x+1$$

Problem 12 (a) For $n = 3$, explain why

$$L_0(x) + L_1(x) + L_2(x) + L_3(x) = 1$$

for all x.
Note: It is unnecessary to actually multiply out and combine the functions $L_i(x)$ of (4.14). The argument you give should generalize easily to higher order interpolation with $n > 3$.
(b) Generalize part (a) to an arbitrary degree $n > 0$.

Solution: (a) $P_3(x) = L_0(x) + L_1(x) + L_2(x) + L_3(x)$ is the unique polynomial of degree at most 3 which interpolates to $f(x) \equiv 1$. $f(x)$ is a polynomial of degree 0, so we must have $f(x) = P_0(x) = P_1(x) = P_2(x) = P_3(x) = 1$.
(b) The proof is similar to the one given in part (a).

Problem 17 Find the solution to the interpolation problem of finding a polynomial $q(x)$ with $\deg(q) \leq 2$ and such that

$$q(x_0) = y_0, \qquad q(x_1) = y_1, \qquad q'(x_1) = y_1'$$

with $x_0 \neq x_1$.
Hint: Write $q(x) = y_0 M_0(x) + y_1 M_1(x) + y_1' M_2(x)$ where $\deg(M_i) \leq 2$, $i = 0, 1, 2$ and each $M_i(x)$ satisfies suitable interpolating conditions at the points x_0 and x_1. For example, $M_0(x)$ should satisfy

$$M_0(x_0) = 1, \qquad M_0(x_1) = M_0'(x_1) = 0$$

Solution: $q(x) = y_0 M_0(x) + y_1 M_1(x) + y_1' M_2(x)$

where

$$\begin{array}{lll} M_0(x_0) = 1 & M_0(x_1) = 0 & M_0'(x_1) = 0 \\ M_1(x_0) = 0 & M_1(x_1) = 1 & M_1'(x_1) = 0 \\ M_2(x_0) = 0 & M_2(x_1) = 0 & M_2'(x_1) = 1 \end{array}$$

Let

$$M_0(x) = a_0(x-x_1)^2 + b_0(x-x_1) + c_0$$

Then

$$M_0(x) = \left(\frac{x-x_1}{x_0-x_1}\right)^2$$

Similarly,

$$M_1(x) = \frac{x-x_0}{x_1-x_0}\left[1 + \frac{x_1-x}{x_1-x_0}\right], \qquad M_2(x) = \frac{x-x_0}{x_1-x_0}(x-x_1)$$

Thus,

$$q(x) = y_0\left(\frac{x-x_1}{x_0-x_1}\right)^2 + y_1\frac{x-x_0}{x_1-x_0}\left[1 + \frac{x_1-x}{x_1-x_0}\right] + y_1'(x-x_1)\frac{x-x_0}{x_1-x_0}$$

Problem 21 (a) Let $f(x)$ be a polynomial of degree m. For $x \neq x_0$, define

$$g_1(x) = f[x_0, x] = \frac{f(x) - f(x_0)}{x - x_0}$$

Show $g(x)$ is a polynomial of degree $m-1$. This is another justification for regarding the divided difference as an analog of the derivative.
Hint: The numerator $f(x) - f(x_0)$ is a polynomial of degree m with a zero at $x = x_0$. Use the fundamental theorem of algebra, noting that x_0 is a root of $f(x) - f(x_0)$.
(b) For x_0, x_1, x distinct and for $f(x)$ a polynomial of degree m, define

$$g_2(x) = f[x_0, x_1, x]$$

Show $g_2(x)$ is a polynomial of degree $m - 2$.

Solution: (a) $f(x) - f(x_0)$ is a polynomial of degree m, and it has a root at $x = x_0$. Therefore, $x - x_0$ is a factor of $f(x) - f(x_0)$, and consequently, $[f(x) - f(x_0)]/(x - x_0)$ will be a polynomial of degree $m - 1$.
The result can also be shown using Taylor's theorem. Using it, $f(x)$ can be written as

$$f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j .$$

Then

$$g_1(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j .$$

(b)

$$g_2(x) = f[x_0, x_1, x] = \frac{f[x_0, x] - f[x_0, x_1]}{x - x_1} = \frac{g_1(x) - g_1(x_1)}{x - x_1} \qquad x \neq x_0, x_1$$

$g_1(x)$ can be written as

$$g_1(x) = b_{m-1}(x - x_1)^{m-1} + \cdots + b_1(x - x_1) + g_1(x_1)$$

So $g_2(x) = b_{m-1}(x - x_1)^{m-2} + \cdots + b_2(x - x_1) + b_1$ which is a polynomial of degree $m - 2$.

Problem 30 (a) In the linear formula (4.33), let $x_1 - x_0 = h$ and $\mu = (x - x_0)/(x_1 - x_0) = (x - x_0)/h$. Show (4.33) reduces to the earlier formula of Problem 4.
(b) In the quadratic formula (4.34), let $x_1 - x_0 = x_2 - x_1 = h$ and $\mu = (x - x_0)/(x_1 - x_0)$. Show that (4.34) reduces to the earlier formula of Problem 9.

Solution: (a)

$$\begin{aligned} P_1(x) &= f(x_0) + (x - x_0) f[x_0, x_1] = f(x_0) + (x - x_0)\frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ &= f(x_0) + \mu[f(x_1) - f(x_0)] \end{aligned}$$

(b)

$$\begin{aligned} P_2(x) &= P_1(x) + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ &= f(x_0) + \mu[f(x_1) - f(x_0)] + \frac{(x - x_0)(x - x_1)}{x_2 - x_0}\left(\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}\right) \end{aligned}$$

$$= f(x_0) + \mu[f(x_1) - f(x_0)] + \frac{(x-x_0)(x-x_1)}{2h^2}\left[(f(x_2) - f(x_1)) - (f(x_1) - f(x_0))\right]$$
$$= f(x_0) + \mu[f(x_1) - f(x_0)] + \frac{1}{2}\mu(\mu-1)\left[f(x_2) - 2f(x_1) + f(x_0)\right]$$

since $\dfrac{x - x_1}{h} = \mu - 1$

4.2

Problem 1 Consider interpolating $\sin x$ from a table of values for $0 \le x \le 1.58$, with the x entries given in steps of $h = 0.01$.
(a) Bound the error of linear interpolation in this table.
(b) Bound the error of quadratic interpolation.

Solution: (a)

$$\sin x - P_1(x) = \frac{(x-x_0)(x-x_1)}{2}(-\sin c_x) \qquad x_0 \le x,\ c_x \le x_1$$

$$|\sin x - P_1(x)| \le \frac{h^2}{8}(1) = 1.25 \times 10^{-5}$$

(b)

$$\sin x - P_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{6}(-\cos c_x) \qquad x_0 \le x,\ c_x \le x_2$$

$$|\sin x - P_2(x)| \le \frac{h^3}{9\sqrt{3}}(1) \doteq 6.42 \times 10^{-8}$$

Problem 3 Repeat Problem 1 with

$$f(x) = \int_0^x e^{-t^2}\,dt$$

for $0 \le x \le 1$

Solution:
(a)

$$f(x) - P_1(x) = \frac{(x-x_0)(x-x_1)}{2}(-2c_x e^{-c_x^2}) \qquad x_0 \le x,\ c_x \le x_1$$

$$|f(x) - P_1(x)| \le \frac{h^2}{8}(\sqrt{2}\,e^{-1/2}) \doteq 1.07 \times 10^{-5}$$

since

$$|f''(x)| = |-2xe^{-x^2}|$$

attains its maximum of $\sqrt{2}\,e^{-1/2}$ at $x = 1/\sqrt{2}$.

(b)

$$f(x) - P_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{6}[2e^{-c_x^2}(2c_x^2-1)] \qquad x_0 \le x,\ c_x \le x_2$$

$$|\ f(x) - P_2(x)\ | \le \frac{h^3}{9\sqrt{3}}(2) = 1.28 \times 10^{-7}$$

since

$$|\ f^{(3)}(x)\ | = |\ 2e^{-x^2}(2x^2-1)\ |$$

attains its maximum of 2 at $x = 0$.

Problem 7 Consider constructing a table of values of $f(x) = \sqrt{x}$ for $1 \le x \le 100$, with values of $f(x)$ given for $x = 0, h, 2h, \ldots$ Choose h so that when linear interpolation is used in this table, the error is bounded by 5×10^{-6}. Discuss and compare the linear interpolation error near $x = 1$ and $x = 100$.

Solution: We have

$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$$

Note that

$$\frac{1}{4}x^{-\frac{3}{2}} \le \frac{1}{4} \qquad x \in [1,\ 100]$$

To have an interpolation error less than 5×10^{-6}, we let

$$|f(x) - P_1(x)| \le \frac{h^2}{8} \cdot \frac{1}{4} \le 5 \times 10^{-6}$$

Then $h \le 0.0126$.

Suppose $h = 0.1$. Then near $x = 1$,

$$|f(x) - P_1(x)| \doteq \frac{(1.005-1)\cdot(1.005-1.01)}{2!} \cdot (-\frac{1}{4} \cdot 1.005^{-\frac{3}{2}}) \doteq 3.1 \times 10^{-6}$$

Near $x = 100$,

$$|f(x) - P_1(x)| \doteq \frac{(99.995-99.990)(99.995-100)}{2!} \cdot (-\frac{1}{4} \cdot 99.995^{-\frac{3}{2}}) \doteq 3.31 \times 10^{-9}$$

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Problem 16 Consider computing the interpolating polynomial $P_n(x)$ for the function

$$f(x) = \frac{1}{2 + \cos x}$$

with a uniformly spaced subdivision of the given interval $[a, b]$. Study the interpolating polynomial and its error for the intervals $[0, 2\pi]$ and $[-\pi, \pi]$, for varying n. Do so for $n = 10, 20, 30, 40$. Make observations on the error.

Hint: Graphs are useful, especially graphs of the interpolation error. Also, double precision arithmetic is insufficient for $n > 40$.

Solution: For the interval $[0, 2\pi]$, the interpolating process doesn't converge to the function $f(x)$, while for the interval $[-\pi, \pi]$, the process does converge as $n \to \infty$.

We list the errors in the table below:

n	10	20	30	40
$[0, 2\pi]$	0.18	0.33	0.82	1.71
$[-\pi, \pi]$	7.5×10^{-3}	1.6×10^{-4}	3.9×10^{-7}	1.8×10^{-7}