ME400/500
HW#7

4.3

Problem 3 Consider the data

x	0	1/2	1	2	3
y	0	1/4	1	−1	−1

(a) Find the piecewise linear interpolating function for the data.
(b) Find the piecewise quadratic interpolating function.
(c) Find the natural cubic spline that interpolates the data.
(d) Find the "not-a-knot" interpolating cubic spline. When using (4.73), let $x_1 = 0$, $x_2 = 1$, $x_3 = 3$, $z_1 = 1/2$, $z_2 = 2$.
Graph all four cases for $0 \le x \le 3$.

Solution: (a)

$$l(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le \frac{1}{2} \\ \frac{3x-1}{2}, & \frac{1}{2} \le x \le 1 \\ -2x+3, & 1 \le x \le 2 \\ -1, & 2 \le x \le 3 \end{cases}$$

(b)

$$q(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ x^2 - 5x + 5, & 1 \le x \le 3 \end{cases}$$

(c) From (4.64) we obtain, after simplification,

$$\begin{aligned} M_1 + 4M_2 + M_3 &= 12 \\ M_2 + 6M_3 + 2M_4 &= -42 \\ M_3 + 4M_4 + M_5 &= 12 \end{aligned}$$

The solution is

$$M_1 = M_5 = 0, \qquad M_2 = M_4 = \frac{38}{7}, \qquad M_3 = \frac{-68}{7}$$

Substituting these values into (4.63) gives

$$s(x) = \begin{cases} \frac{38}{21}x^3 + \frac{1}{21}x, & 0 \le x \le \frac{1}{2} \\ -\frac{106}{21}x^3 + \frac{72}{7}x^2 - \frac{107}{21}x + \frac{6}{7}, & \frac{1}{2} \le x \le 1 \\ \frac{53}{21}x^3 - \frac{87}{7}x^2 + \frac{370}{21}x - \frac{47}{7}, & 1 \le x \le 2 \\ -\frac{19}{21}x^3 + \frac{57}{7}x^2 - \frac{494}{21}x + \frac{145}{7}, & 2 \le x \le 3 \end{cases}$$

(d) Using (4.64),

$$\frac{1}{6}M_1 + M_2 + \frac{1}{3}M_3 = -2$$

From (4.73)

$$M_1 + M_2 = 4$$

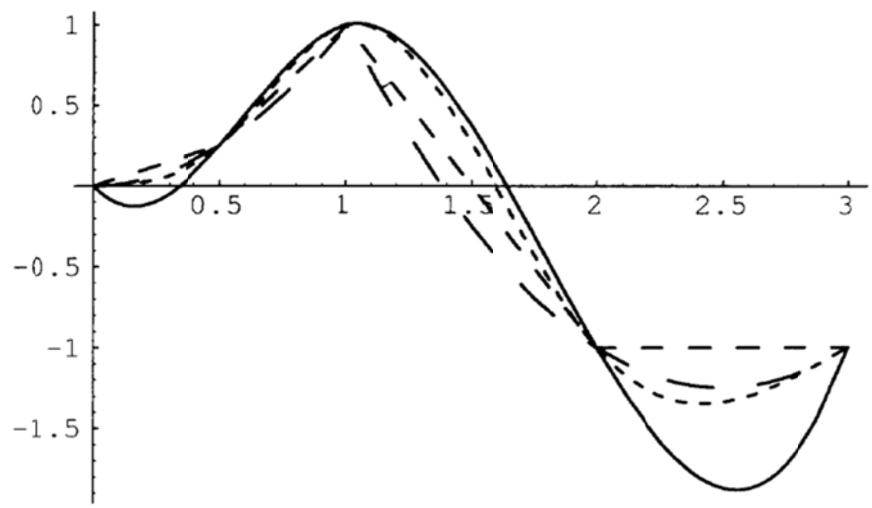
$$M_2 + M_3 = 4$$

So

$$M_1 = M_3 = 12, \qquad M_2 = -8$$

Substituting these values into (4.63), we obtain

$$s(x) = \begin{cases} \dfrac{-10x^3+18x^2-5x}{3}, & 0 \le x \le 1 \\ \dfrac{5x^3-27x^2+40x-15}{3}, & 1 \le x \le 3 \end{cases}$$



small dash: (c) Natural cubic spline median dash: $l(x)$ big dash: $q(x)$
solid line: (d) "not-a-knot" cubic spline

Problem 7 (a) Solve for the cubic spline that interpolates the data in (4.66), with the addition of the boundary conditions (4.71),

$$s'(1) = -1, \qquad s'(4) = -\frac{1}{16}$$

Hint: Combine (4.74) in problem 6 with (4.67), and use a linear system solver to solve your linear system of four equations.
(b) Compare both the natural spline (4.68) and the present spline to the function $f(x) = 1/x$ from which the data were generated. Calculate $f(x) - s(x)$ for a sampling of points in $1 \le x \le 4$.

Solution: (a) After multiplying each equation by 6, the linear system is

$$\begin{aligned} 2M_1 + M_2 &= 3 \\ M_1 + 4M_2 + M_3 &= 2 \\ M_2 + 4M_3 + M_4 &= \frac{1}{2} \\ M_3 + 2M_4 &= \frac{1}{8} \end{aligned}$$

The first and last equations arose from (4.79). The solution is

$$M_1 = \frac{173}{120}, \qquad M_2 = \frac{7}{60}, \qquad M_3 = \frac{11}{120}, \qquad M_4 = \frac{1}{60}$$

Using (4.63)

$$s(x) = \begin{cases} \frac{-53x^3+332x^2-745x+706}{240}, & 1 \le x \le 2 \\ \frac{-x^3+20x^2-121x+290}{240}, & 2 \le x \le 3 \\ \frac{-3x^3+38x^2-175x+344}{240}, & 3 \le x \le 4 \end{cases}$$

(b)

x	Natural spline $s_n(x)$	Error $\frac{1}{x} - s_n(x)$	part(a) spline $s(x)$	Error $\frac{1}{x} - s(x)$
1.25	0.85547	−5.55E−2	0.79160	8.40E−3
1.50	0.71875	−5.21E−2	0.65260	1.41E−2
1.75	0.59766	−2.62E−2	0.56230	9.12E−3
2.25	0.43099	1.35E−2	0.44837	−3.93E−3
2.50	0.38542	1.46E−2	0.40365	−3.65E−3
2.75	0.35547	8.17E−3	0.36543	−1.79E−3
3.25	0.31250	−4.81E−3	0.30684	8.56E−4
3.50	0.29167	−5.95E−3	0.28490	8.18E−4
3.75	0.27083	−4.17E−3	0.26634	3.26E−4

The spline from part(a) with the endpoint derivative conditions performs better than the natural spline. The reason is that $f''(x) = 2/x^3 \neq 0$.

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4.4

Problem 2 (a) For $f(x) = \tan^{-1} x$, calculate the Taylor approximations $t_1(x)$ and $t_3(x)$. Also find their maximum error relative to $\tan^{-1} x$ on $[-1, 1]$.
(b) The linear and cubic minimax polynomials for $f(x) = \tan^{-1} x$ on $[-1, 1]$ are, respectively,

$$m_1(x) = 0.833278x$$
$$m_3(x) = 0.97238588x - 0.19193797x^3$$

Find their maximum errors on $[-1, 1]$.
(c) Graph $f(x) - t_3(x)$ and $f(x) - m_3(x)$ on $[-1, 1]$.

Solution: (a) Recall from Chapter 1 that

$$\tan^{-1} x = \sum_{j=0}^{n} (-1)^j \frac{x^{2j+1}}{2j+1} + \frac{(-1)^n}{1+c_x^2} \cdot \frac{x^{2n+3}}{2n+3}$$

Then

$$t_1(x) = x, \qquad t_3(x) = x - \frac{x^3}{3}$$

$$\max_{-1\le x\le 1} |\tan^{-1} x - t_1(x)| = 0.215 \text{ at } x = \pm 1$$

$$\max_{-1\le x\le 1} |\tan^{-1} x - t_3(x)| = 0.119 \text{ at } x = \pm 1$$

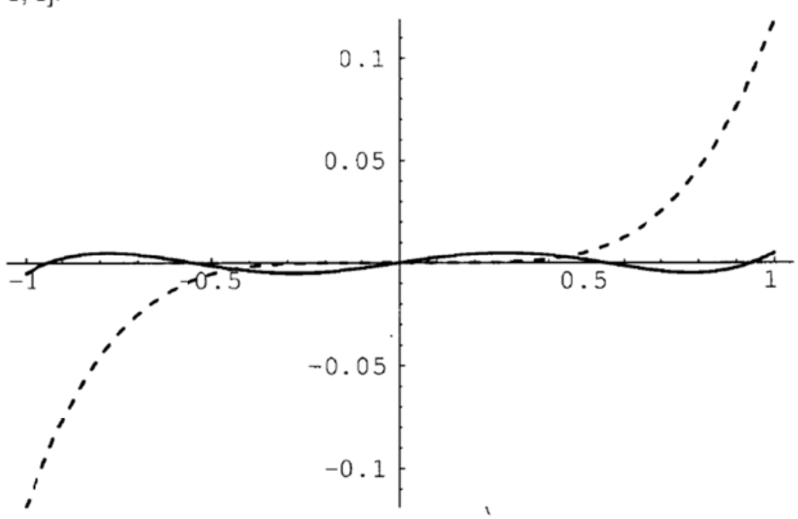
(b) By direct computation at a large number of points in $[-1, 1]$,

$$\max_{-1\le x\le 1} |\tan^{-1} x - m_1(x)| \doteq 0.0479,$$

at four points in $[-1, 1]$.

$$\max_{-1\le x\le 1} |\tan^{-1} x - m_3(x)| \doteq 0.00495,$$

at six points in $[-1, 1]$.
(c)



dash: $\tan^{-1} x - t_3(x)$ solid: $\tan^{-1} x - m_3(x)$

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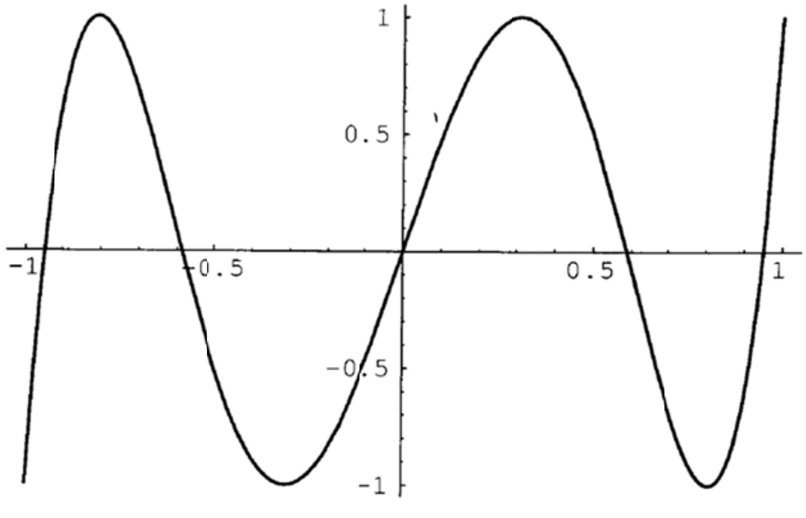
4.5

Problem 1 Find $T_5(x)$ explicitly in polynomial form, and then graph it on $-1 \le x \le 1$.

Solution: Use the triple recursion relation (4.87) and the formula for $T_3(x)$ and $T_4(x)$ in this section. Then

$$T_5(x) = 2x\left(8x^4 - 8x^2 + 1\right) - \left(4x^3 - 3x\right) = 16x^5 - 20x^3 + 5x$$



$y = T_5(x)$

4.6

Problem 2 Give the interpolation nodes for the linear near-minimax approximation of this section, for the interval $[-1, 1]$. Give the linear near-minimax approximation for $f(x) = e^x$ on $[-1, 1]$.

Solution: $n = 1$, and the nodes are the zeros of $T_2(x)$. These are

$$x_1 = \frac{1}{\sqrt{2}}, \qquad x_2 = \frac{-1}{\sqrt{2}}$$

Then

$$P_1(x) = f\left(\frac{-1}{\sqrt{2}}\right) + \frac{x + \frac{1}{\sqrt{2}}}{\sqrt{2}}\left(f\left(\frac{1}{\sqrt{2}}\right) - f\left(\frac{-1}{\sqrt{2}}\right)\right)$$

$$= e^{\frac{-1}{\sqrt{2}}} + \frac{x + \frac{1}{\sqrt{2}}}{\sqrt{2}}\left(e^{\frac{1}{\sqrt{2}}} - e^{\frac{-1}{\sqrt{2}}}\right) \doteq 1.0854x + 1.2606$$

$$\max_{-1 \le x \le 1} |\, e^x - P_1(x) \,| = 0.372$$

Problem 4 Most functions do not have $[-1, 1]$ as the interval on which they are to be approximated. Suppose $g(t)$ is to be evaluated for $a \le t \le b$. Then define a new function $f(x)$ on $[-1, 1]$ by

$$f(x) = g\left(\frac{(b+a)+x(b-a)}{2}\right), \qquad -1 \le x \le 1$$

Here

$$t = \frac{1}{2}[(b+a)+x(b-a)]$$

represents a linear change of variable. We now approximate $f(x)$ on $[-1, 1]$.

As a specific example, produce the cubic near-minimax approximation for $g(t) = e^t$, $0 \le t \le 1$. Compare this to the minimax approximation given in Problem 3(b) of Section 4.4.

Solution: Let $[a, b] = [0, 1]$, and consider $P_3(x) \doteq e^t$, $0 \le t \le 1$, with $t = (x+1)/2$, $-1 \le x \le 1$. Then the near-minimax $P_3(x)$ can be constructed from the following set of nodes and corresponding divided differences, as in the program in the text.

j	x_j	$D_j f$
0	0.923880	2.616767
1	0.382683	1.146304
2	−0.382683	0.242548
3	−0.923880	0.034780

4.7

Problem 2 Find the linear least squares approximation to $f(x) = \sin x$ on the interval $[0, \frac{1}{2}\pi]$. Use the direct method of Example 4.7.1, but on the interval $[0, \frac{1}{2}\pi]$.

Solution: Let $f(x) = \sin x$, and let $p(x) = \alpha_0 + \alpha_1 x$.

$$g(\alpha_0, \alpha_1) = \int_0^{\pi/2} [\sin x - \alpha_0 - \alpha_1 x]^2 dx$$

From $\frac{\partial g}{\partial \alpha_0} = \frac{\partial g}{\partial \alpha_1} = 0$, we obtain:

$$2\int_0^{\pi/2} (\sin x - \alpha_0 - \alpha_1 x)(-1)dx = 0$$

$$2\int_0^{\pi/2} (\sin x - \alpha_0 - \alpha_1 x)(-x)dx = 0$$

These simplifies to

$$\begin{cases} \frac{\pi}{2}\alpha_0 + \frac{\pi^2}{8}\alpha_1 = \int_0^{\pi/2} \sin x dx \\ \frac{\pi^2}{8}\alpha_0 + \frac{\pi^3}{24}\alpha_1 = \int_0^{\pi/2} x \sin x dx \end{cases}$$

So $\alpha_0 \doteq 0.1148$, $\alpha_1 \doteq 0.6644$, and the approximation is $p(x) \doteq 0.1148 + 0.6644x$.