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5.1

Problem 4 Use the trapezoidal rule and Simpson's rule with $n = 4, 8, \ldots, 512$ to find approximate values of the area under the curve of $y = f(x)$ for the following functions $f(x)$ on the given intervals:
(a) $f(x) = e^{-x^2}, \quad 0 \le x \le 10$
(b) $f(x) = \tan^{-1}\left(1 + x^2\right), \quad 0 \le x \le 2$
(c) $f(x) = \sqrt{x}e^x, \quad 0 \le x \le 1$

Solution: Use the program for the trapezoidal rule given in the text.
(a)

n	T_n
4	1.2548261354D+00
8	8.8942827806D−01
16	8.8622692547D−01
32	8.8622692545D−01
64	8.8622692545D−01
128	8.8622692545D−01
256	8.8622692545D−01
512	8.8622692545D−01

(c)

n	T_n
4	1.2500878519D+00
8	1.2516121252D+00
16	1.2536843988D+00
32	1.2548090999D+00
64	1.2553062340D+00
128	1.2555071316D+00
256	1.2555844895D+00
512	1.2556134304D+00

Problem 12 Determine the degree of precision of the approximation

$$\int_0^1 f(x)\,dx \approx \frac{1}{4}f(0) + \frac{3}{4}f\left(\frac{2}{3}\right)$$

Solution: Degree of precision is two.
Let $k(f) = 1/4 * f(0) + 3/4 * f(2/3)$
For $f(x) = 1, \quad k(f) = 1, \quad I(f) = 1, \quad I(f) - k(f) = 0$
For $f(x) = x, \quad k(f) = \frac{1}{4} * 0 + \frac{3}{4} * \frac{2}{3} = \frac{1}{2}, \quad I(f) = \frac{1}{2}, \quad I(f) - k(f) = 0$
For $f(x) = x^2, \quad k(f) = \frac{1}{4} * 0 + \frac{3}{4} * (\frac{2}{3})^2 = \frac{1}{3}, \quad I(f) = \frac{1}{3}, \quad I(f) - k(f) = 0$
For $f(x) = x^3, \quad k(f) = \frac{1}{4} * 0 + \frac{3}{4} * (\frac{2}{3})^3 = \frac{2}{9}, \quad I(f) = \frac{1}{4}, \quad I(f) - k(f) = \frac{1}{36}$
So the degree of precision is two.

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Problem 15 Consider the approximation

$$I(f) = \int_{-1}^{1} f(x)\,dx \approx f(-\beta) + f(\beta)$$

for some β satisfying $0 < \beta \leq 1$. Show it has degree of precision greater than or equal to 1 for any such choice of β. Choose β to obtain a formula with degree of precision greater than 1. What is the degree of precision of this formula?

Solution: For $\beta = 3^{-1/2}$, the resulting numerical integration formula is $I_\beta(f) = f(-3^{-1/2}) + f(3^{-1/2})$. Let us show the degree of precision of this formula is three. It is easy to verify that $I(f) - I_\beta(f) = 0$ for $f(x) = 1$, x, x^2, x^3. For $f(x) = x^4$,

$$I(f) - I_\beta(f) = \frac{8}{45}$$

For any β satisfying $0 < \beta \leq 1$, for $f(x) = 1$ and x, we have $I(f) - I_\beta(f) = 0$. So the formula has a degree of precision at least 1.

5.2

Problem 1 Using the error formula (5.23), bound the error in $T_n(f)$ applied to the following integrals.

(a) $\int_0^{\pi/2} \cos x\,dx$ (b) $\int_0^1 e^{-x^2}\,dx$ (c) $\int_0^{\sqrt{\pi}} \cos(x^2)\,dx$

Solution: (a) $f(x) = \cos x$, $f''(x) = -\cos x$, $b - a = \pi/2$

$$E_n^T(f) = \frac{-h^2}{12}\left(\frac{\pi}{2}\right)(-\cos c_n) \qquad |\,E_n^T(f)\,| \leq \frac{\pi h^2}{24}$$

(b) $f(x) = e^{-x^2}$, $f''(x) = 2e^{-x^2}(2x^2 - 1)$, $b - a = 1$

$$\max_{0 \leq x \leq 1} |\,f''(x)\,| = 2, \text{ occurring at } x = 0.$$

$$|\,E_n^T(f)\,| \leq \frac{h^2}{12}(2) = \frac{h^2}{6}$$

Problem 7 (a) Consider using the trapezoidal rule T_n to estimate the integral

$$I = \int_1^3 \log x\,dx$$

Give both a rigorous error bound for $I - T_n$ and an asymptotic error bound for $I - T_n$. Using the rigorous error bound, say how large n should be chosen in order that $|I - T_n| \leq 5 \times 10^{-8}$.

(b) Repeat with Simpson's rule.

Solution: (a) The rigorous error bound for $I - T_n$.

$$f(x) = \log x, \quad f''(x) = \frac{-1}{x^2}, \quad b - a = 2$$

$$\max_{1 \leq x \leq 3} |\,f''(x)\,| = 1, \text{ occurring at } x = 1.$$

$$| E_n^T(f) | \le 2\frac{h^2}{12} = \frac{h^2}{6}, \qquad h = \frac{2}{n}$$

$$| E_n^T(f) | \le 5 * 10^{-8} \qquad \text{if } n \ge 366$$

The asymptotic error bound for $I - T_n$.

$$\tilde{E}_n^T(f) = \frac{-h^2}{12}(f'(3) - f'(1)) = \frac{h^2}{18}, \qquad h = \frac{2}{n}$$

(b) The rigorous error bound for $I - S_n$.

$$f(x) = \log x, \quad f^{(4)}(x) = \frac{-6}{x^4}, \quad b - a = 2$$

$$\max_{1 \le x \le 3} | f^{(4)}(x) | = 6, \quad \text{occurring at } x = 1.$$

$$| E_n^S(f) | \le \frac{h^4}{15}, \qquad h = \frac{2}{n}$$

$$| E_n^S(f) | \le 5 * 10^{-8} \qquad \text{if } n \ge 68$$

The asymptotic error bound for $I - S_n$.

$$\tilde{E}_n^S(f) = \frac{-h^4}{180}\left(f^{(3)}(3) - f^{(3)}(1)\right) = \frac{13}{1215}h^4, \qquad h = \frac{2}{n}$$

Problem 17 Use Richardson extrapolation to estimate the errors in Problems 2(a), (b), (c), and (e) of Section 5.1.

(c)

n	True Error	Estimate
2	1.73E−01	
4	7.11E−02	3.40E−02
8	7.50E−03	2.12E−02
16	1.95E−03	1.85E−03
32	4.89E−04	4.88E−04
64	1.22E−04	1.22E−04
128	3.06E−05	3.06E−05

Problem 18 Use Richardson extrapolation to estimate the errors in Problems 3(a), (b), (c), and (e) of Section 5.1.

Solution:

(c)

n	True Error	Estimate
2	−2.85E−01	
4	3.71E−02	−2.15E−02
8	−1.37E−02	3.39E−03
16	1.06E−04	−9.21E−04
32	1.08E−06	6.99E−06
64	6.74E−08	6.75E−08
128	4.22E−09	4.22E−09

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Problem 20 (a) Apply Simpson's rule to $I = \int_0^1 \sin\sqrt{x}\,dx$ with $n = 2, 4, 8, \ldots, 128$. Use problem 13(b) to calculate the rate of convergence.
(b) Transform I using the change of variable $x = t^2$, obtaining

$$I = 2\int_0^1 t\sin t\,dt\,.$$

Apply Simpson's rule to this new integral with $n = 2, 4, 8, \ldots, 128$, and compare the results with those (a).

Solution: The true answer is

$$I = 2[\sin 1 - \cos 1] \doteq 0.6023373578795135785$$

In the following tables,

$$Error = I_{2n} - I_n\,, \qquad Ratio = \frac{I_{2n} - I_n}{I_{4n} - I_{2n}}$$

(a)

n	S_n	Error	Ratio
2	0.57333645685		
4	0.59212391192	1.88E−02	30.5
8	0.59873699516	6.61E−03	2.84
16	0.60106658484	2.33E−03	2.84
32	0.60188846846	8.22E−04	2.83
64	0.60217872264	2.90E−04	2.83
128	0.60228128449	1.03E−04	2.83
256	0.60231753518	3.63E−05	2.83
512	0.60233034989	1.28E−06	2.83

This shows a rate of convergence of $O(h^{1.5})$. The slow rate is due to the fact that the integrand behaves like $\sqrt{x}$ for small values of x.
(b)

n	S_n	Error	Ratio
2	0.60010735401		
4	0.60220279379	2.10E−03	
8	0.60232902030	1.26E−04	16.6
16	0.60233683791	7.82E−06	16.1
32	0.60233732540	4.87E−07	16.0
64	0.60233735585	3.05E−08	16.0
128	0.60233735775	1.90E−09	16.0
256	0.60233735787	1.19E−10	16.0
512	0.60233735788	7.43E−12	16.0

This shows a rate of convergence of $O(h^4)$. The reason is that the integrand is sufficiently differentiable.