ME-500: Solution to Sample Exam 2 (Fall 2012)

1. a)

$$L_2^0(x) = \frac{(x-180)(x-200)}{(160-180)(160-200)} = 0.00125x^2 - 0.475x + 45$$

$$L_2^1(x) = \frac{(x-160)(x-200)}{(180-160)(180-200)} = -0.0025x^2 + 0.9x - 80$$

$$L_2^1(x) = \frac{(x-160)(x-180)}{(200-160)(200-180)} = 0.00125x^2 - 0.425x + 36$$

$$p_2(x) = L_2^0(x)\times 7 + L_2^1(x)\times 5.5 + L_2^2(x)\times 5 = 0.00125x^2 - 0.5x = 55$$

b)

$$p_1(x) = \frac{(180-x)\times 7 + (x-160)\times 5.5}{20} = -0.75x + 19 \qquad 160 \le x \le 180$$

$$p_2(x) = \frac{(200-x)\times 5.5 + (x-180)\times 5}{20} = -0.025x + 10 \qquad 180 \le x \le 200$$

c) On each subinterval $E = f(x) - p(x) = \dfrac{(x-x_0)(x-x_1)}{2} f''(c_x)$

Using the problem data the second derivative can be approximated by

$$f''(x) \approx \frac{7 - 2\times 5.5 + 5}{20^2} = -0.0025 \text{ therefore } \quad E \approx \left|\frac{h^2}{8}(-0.0025)\right| \doteq 0.125$$

2. a)

$$I \cong \frac{h}{2}\big(f(x_0) + 2f(x_1) + f(x_2)\big) = \frac{1}{4}\left(\frac{1}{2} + \frac{2}{1.5} + 1\right) \doteq 0.708333$$

b)

$$E = -\frac{h^2(b-a)}{12} f''(c) \quad , \quad f''(x) = \frac{-2}{(2-x)^3} \;\Rightarrow\; \max|f''(x)| = 2$$

$$|E| \le \frac{h^2 \times 2}{12} \times 1 < 5\times 10^{-6} \quad \Rightarrow \quad h \le 0.005477$$

c)

$$I \cong \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2) \right) = \frac{1}{6}\left(\frac{1}{2} + \frac{4}{1.5} + 1 \right) \doteq 0.694444$$

d)

$$I_2 = hf\left((x_0 + x_1)/2 \right) + hf\left((x_1 + x_2)/2 \right) = \frac{1}{2}\left(\frac{1}{1.75} \right) + \frac{1}{2}\left(\frac{1}{1.25} \right) \doteq 0.685714$$

e)

$I_1 = 1 \times \frac{1}{1.5} \doteq 0.666667$ So applying Richardson's extrapolation formula

$$I \cong \frac{1}{2^2 - 1}\left[2^2 I_2 - I_1 \right] = \frac{1}{3}(4 \times 0.685714 - 0.6666667) = 0.692063$$

3.

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + Higher\,Order Terms$$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2} f''(x) + \frac{8h^3}{6} f'''(x) + HOT$$

$$Af(x-h) + Bf(x+2h) = (A+B)f(x) + h(-A+2B)f'(x) + \frac{h^2}{2}(A+4B)f''(x) + HOT$$

We must set $A + B = 0$ and $h(-A + 2B) = 1$. This gives $A = -\frac{1}{3h}$, $B = \frac{1}{3h}$. Hence

$$f'(x) \cong \frac{f(x+2h) - f(x-h)}{3h} + \frac{h}{2} f''(x) + HOT$$