THE PETROV-GALERKIN METHOD

Consider the Galerkin solution using Linear elements of the modified convection-diffusion equation

$$-\left(D+\frac{\alpha u h}{2}\right)\frac{d^2\phi}{dx^2}+u\frac{d\phi}{dx}=0$$

α is a parameter between 0 and 1. If $\alpha=0$, we will have the discrete Galerkin form of the convective-diffusion equation. If $\alpha=1$ we recover the Upwind form.

Because we expect that there is a value of α such that the term $\alpha uh/2$ gives us the amount of numerical diffusion necessary to obtain the correct answer, we call this term the "*Balancing Diffusion*"

The Galerkin discretization using Linear elements of uniform size $\Delta x = h$ produces the difference equations

$$\left[1+\frac{\gamma}{2}(\alpha+1)\right]\phi_{i-1}-2\left(1+\frac{\alpha\gamma}{2}\right)\phi_i+\left[1+\frac{\gamma}{2}(\alpha-1)\right]\phi_{i+1}$$

The solution is $\phi_i = A + B\left[\frac{2+\gamma(\alpha+1)}{2+\gamma(\alpha-1)}\right]^i$

It follows that the solution is non-oscillatory if $\alpha \geq 1-\frac{\gamma}{2}$.

We define the "*Critical Value*" $\alpha_{cr}=1-\frac{\gamma}{2}$.

We define the Truncation Error TE as the difference between the original differential equation and the modified discrete form, i.e.

$$TE=\left[-\frac{d^2\phi}{dx^2}+\frac{\gamma}{h}\frac{d\phi}{dx}\right]_{x=x_i}-\frac{1}{h^2}\left\{\left[1+\frac{\gamma}{2}(\alpha+1)\right]\phi_{i-1}-2\left(1+\frac{\alpha\gamma}{2}\right)\phi_i+\left[1+\frac{\gamma}{2}(\alpha-1)\right]\phi_{i+1}\right\}$$

Keeping track of derivatives up to order 8, we get

$$TE \equiv \frac{\alpha\gamma}{2}\phi_i^{(2)}+\frac{2}{h^2}\left(1+\frac{\alpha\gamma}{2}\right)\left[\frac{h^4}{4!}\phi_i^{(4)}+\frac{h^6}{6!}\phi_i^{(6)}+\frac{h^8}{8!}\phi_i^{(8)}+\ldots\right]-\frac{\gamma}{h^2}\left[\frac{h^3}{3!}\phi_i^{(3)}+\frac{h^5}{5!}\phi_i^{(5)}+\frac{h^7}{7!}\phi_i^{(7)}+\ldots\right]$$

Now we use the differential equation written as $\phi_i^{(2)}=\frac{\gamma}{h}\phi_i^{(1)}$ to write all higher order derivatives in terms of the second derivative. That is we get the recursive relation $\phi_i^{(n)}=\left(\frac{\gamma}{h}\right)^{n-2}\phi_i^{(2)}$

Substituting into the truncation error we get

$$TE=\left\{\frac{\alpha\gamma}{2}+2\left(1+\frac{\alpha\gamma}{2}\right)\left[\frac{\gamma^2}{4!}+\frac{\gamma^4}{6!}+\frac{\gamma^6}{8!}+\ldots\right]-\gamma\left[\frac{\gamma}{3!}+\frac{\gamma^3}{5!}+\frac{\gamma^5}{7!}+\ldots\right]\right\}\phi_i^{(2)}$$

The total truncation error can be expressed in terms of a numerical diffusion that takes the form

$$TE=\left[\frac{1}{\gamma^2}\left\{2\left(1+\frac{\alpha\gamma}{2}\right)\tanh\left(\frac{\gamma}{2}\right)-\gamma\right\}\sinh(\gamma)\right]\phi_i^{(2)}$$

Set $f(\gamma)=\frac{1}{\gamma^2}\left\{2\left(1+\frac{\alpha\gamma}{2}\right)\tanh\left(\frac{\gamma}{2}\right)-\gamma\right\}\sinh(\gamma)$

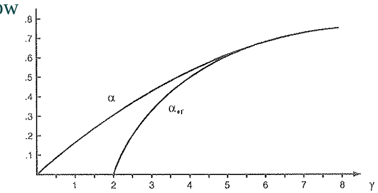
1) Set $\alpha=0$, and re-write $f(\gamma)$ as $f(\gamma)=\left(\frac{\tanh(\gamma/2)}{(\gamma/2)}-1\right)\frac{\sinh(\gamma)}{\gamma}$.

Because $\frac{\sinh(\gamma)}{\gamma}>0$ and $\left(\frac{\tanh(\gamma/2)}{(\gamma/2)}-1\right)<0$ for all γ, $f(\gamma)<0$ for all γ.

Therefore, the Galerkin method is always *underdiffused*.

2) Set $TE=0$ and solve for α, we get $\alpha=\coth\left(\frac{\gamma}{2}\right)-\frac{1}{\gamma}$

This value of α is called the optimal value α_{opt}. It produces the exact solution when the coefficients are constant and superconvergent solutions in the general case. α_{opt} and α_{cr} are shown below

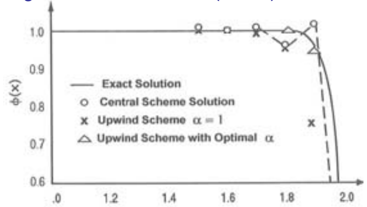


EXAMPLE

Use 10 Linear elements to solve the equation

$$-\frac{d^2\phi}{dx^2}+\frac{60}{x}\frac{d\phi}{dx}=0,\ 1<x<2,\ \phi(1)=1,\ \phi(2)=0$$

Solution using 10 Linear elements (*h*=0.1)



Solution is exact to 4 significant digits, the average velocity was used in each element

Evaluating α involves a coth, and as seen in the previous figure α_{opt} and α_{cr} are very close after $\gamma>5$. The amount of additional numerical diffusion added using α_{cr} instead of α_{opt} at $\gamma=8$ is less than 1%.The same is true if we use $\alpha=0$ when $\gamma<0.1$. So α is calculated from

$$\alpha=\begin{cases}0.0 & if\ \ \gamma<0.1\\ \coth(\gamma/2)-1/\gamma & if\ \ 0.1\leq\gamma\leq 8.0\\ 1-2/\gamma & if\ \ \gamma>8.0\end{cases}$$

The concept of Balancing Diffusion, although effective in simple situations is difficult to use in more complex problems. We will re-cast the ideas under the framework of a **PETROV-GALERKIN** method that can be extended to all cases in 1-, 2-, and 3-dimensions.

The name **Petrov-Galerkin** refers to a method in which the weighting functions are NOT the same as the shape functions in a Galerkin formulation.

Lets go back to the weighted residual form of the modified convective-diffusion equation ignoring the boundary terms that are not important in this discussion. We have

$$\int_0^L \left[\left(D + \frac{\alpha u h}{2} \right) \frac{dw}{dx}\frac{d\phi}{dx} + wu\frac{d\phi}{dx} \right] dx = 0$$

The Galerkin formulation is (using linear shape functions)

$$\int_0^L \left[\left(D + \frac{\alpha u h}{2} \right) \frac{dN_i}{dx}\frac{d\phi}{dx} + N_i u\frac{d\phi}{dx} \right] dx = 0, \text{ where } \phi = N_1\phi_1 + N_2\phi_2$$

Then re-write the integral in the form

$$\int_0^L \left[D\frac{dN_i}{dx}\frac{d\phi}{dx} + u\left(N_i + \frac{\alpha h}{2}\frac{dN_i}{dx} \right)\frac{d\phi}{dx} \right] dx = 0$$

7

Now we define $w_i = N_i + \frac{\alpha h}{2}\frac{dN_i}{dx}$, then the weak form becomes

$$\int_0^L \left[D\frac{dw_i}{dx}\frac{d\phi}{dx} + w_i u\frac{d\phi}{dx} \right] dx = 0$$

This is the basic Petrov-Galerkin method for the one-dimensional Convection –Diffusion equation.

Petrov-Galerkin methods using quadratic elements have also been built. In this case two parameters α and β are needed due to the different nature of the internal node. Here we will restrict ourselves to the use of linear elements.

Introduce a non-zero source term in the equation.

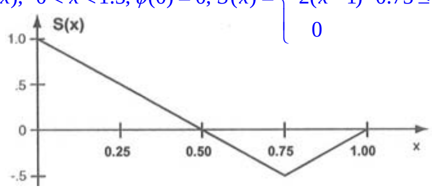
$$-\frac{d}{dx}\left(D\frac{d\phi}{dx} \right) + u\frac{d\phi}{dx} = S(x)$$

The weighted residual form is $\int_0^L \left(D\frac{dw}{dx}\frac{d\phi}{dx} + uw\frac{d\phi}{dx} \right) dx = \int_0^L wSdx$

8

The following example illustrates how the Petrov-Galerkin method Automatically extends consistently to treat other terms in the Equation. We set D = 0 and look at the equation

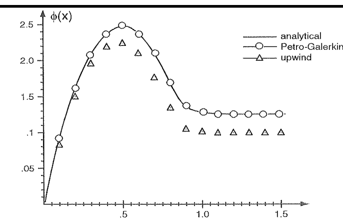
$$\frac{d\phi}{dx} = S(x),\ \ 0 < x < 1.5,\ \phi(0) = 0,\ S(x) = \begin{cases} 1 - 2x & 0 \le x \le 0.75 \\ 2(x-1) & 0.75 \le x \le 1.0 \\ 0 & x > 0 \end{cases}$$



Two numerical solutions with h = 0.1 are shown in the next slide.

1) Using added balancing diffusion, which does nothing to the right hand side.
2) Petrov-Galerkin, which consistently weights the right hand side.

9



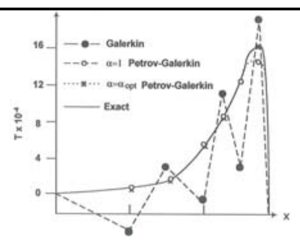
Note that in this problem $\gamma = \infty$, therefore full upwinding $(\alpha = 1)$ is required.

In page 329 in the text it is shown that the Petrov-Galerkin solution is exact if the source term is piecewise linear and nodes are placed at the points of slope discontinuity.

EXAMPLE

$$-\frac{d^2T}{dx^2} + 200\frac{dT}{dx} = x^2,\ \ 0 < x < 1,\ T(0) = T(1) = 0$$

10



x	Analytical Solution	Regular mesh h = 0.05			Irregular mesh		
		α = 0	α = 1	P-G	α = 0	α = 1	P-G
0.1	0.00000	0.00000	0.00000	0.00000	-	-	-
0.3	0.00005	0.00004	0.00005	0.00005	-0.00062	0.00004	0.00004
0.4	0.00011	0.00010	0.00011	0.00011	-	-	-
0.5	0.00021	0.00018	0.00021	0.00021	-	-	-
0.55	0.00028	0.00033	0.00028	0.00028	0.00031	0.00029	0.00029
0.6	0.00037	0.00030	0.00037	0.00037	-	-	-
0.7	0.00058	0.00043	0.00058	0.00058	-0.00014	0.00059	0.00059
0.8	0.00087	0.00053	0.00087	0.00087	0.00105	0.00087	0.00087
0.9	0.00124	0.00048	0.00122	0.00124	0.00031	0.00123	0.00124
0.95	0.00145	0.000258	0.00130	0.00145	0.00206	0.00131	0.00146

11

Reaction – Diffusion Equations

$$-D\frac{d^2\phi}{dx^2} + u\frac{d\phi}{dx} + a\phi = S$$

When the reaction term $a\phi$ dominates, these equations also experienece numerical oscillations. Stable solutions can be obtain by means of two stabilizing parameters, but it becomes more difficult to determine them. References to the basic work are given in the text, page 331.

THE PETROV – GALERKIN METHOD IN TWO DIMENSIONS

The basic equation, $-\left(\frac{\partial}{\partial x}(D\frac{\partial\phi}{\partial x}) + \frac{\partial}{\partial y}(D\frac{\partial\phi}{\partial y}) \right) + u\frac{\partial\phi}{\partial x} + \mathrm{v}\frac{\partial\phi}{\partial y} = 0$

will be written as $-\nabla\mathbf{D}\nabla\phi + \mathbf{V}\cdot\nabla\phi = \mathbf{0}$

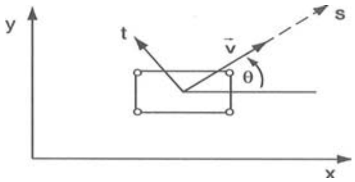
where $\nabla_{xy} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$, $\mathbf{D} = D\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{V} = \begin{bmatrix} u \\ \mathrm{v} \end{bmatrix}$

12

We assume that the velocity vector is constant for convenience.

This can be for example the average fluid velocity in the element.

Consider a rectangular bilinear element as shown in the figure



We introduce a local (element – wise) rotation of the operator to a new coordinate system *s* –*t* such that *s* is aligned in the direction of the velocity vector. In this new coordinate system the equation becomes

$$-\nabla_{st}^T \mathbf{D} \nabla_{st} \phi + |\mathbf{V}| \frac{\partial \phi}{\partial s} = 0$$

13

$|\mathbf{V}|$ is the magnitude of the velocity and the gradient $\nabla_{st} = \mathbf{T}\nabla_{xy}$

where $\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is the rotation matrix.

Now re-write the equation in the form

$$-\frac{\partial}{\partial s}\left(D\frac{\partial \phi}{\partial s}\right) + |\mathbf{V}|\frac{\partial \phi}{\partial s} = \frac{\partial}{\partial t}\left(D\frac{\partial \phi}{\partial t}\right)$$

This equation can be viewed as a one-dimensional convection-diffusion equation in the *s* – direction, with a source term.

From this point of view, we must introduce abalancing diffusion only in the *s* – direction. That is an "*anisotropic balancing diffusion*". Given by

$$\mathbf{D}^1 = D^1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{\alpha|\mathbf{V}|\bar{h}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

REMARKS: 1) $\bar{h}$ is an average element length defined later.

2) α is calculated as before using $\gamma = \frac{|\mathbf{V}|\bar{h}}{D}$

14

The modified equation, after adding the balancing diffusion is

$$\nabla^T(\mathbf{D} + \mathbf{D}^1)\nabla\phi + |\mathbf{V}|\frac{\partial \phi}{\partial s} = 0$$

Rotating back to the x - y system we get

$$-\left\{\frac{\partial}{\partial x}\left(D\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial x}\left[\frac{\alpha u \bar{h}}{2}\left(\frac{u}{|\mathbf{V}|}\frac{\partial \phi}{\partial x} + \frac{\mathrm{v}}{|\mathbf{V}|}\frac{\partial \phi}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left(D\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial y}\left[\frac{\alpha \mathrm{v} \bar{h}}{2}\left(\frac{u}{|\mathbf{V}|}\frac{\partial \phi}{\partial x} + \frac{\mathrm{v}}{|\mathbf{V}|}\frac{\partial \phi}{\partial y}\right)\right]\right\} + u\frac{\partial \phi}{\partial x} + \mathrm{v}\frac{\partial \phi}{\partial y} = 0$$

The weak Galerkin formulation is written as

$$\int_\Omega \left\{\frac{\partial N_i}{\partial x}\left[D\frac{\partial \phi}{\partial x} + \frac{\alpha u \bar{h}}{2|\mathbf{V}|}\left(u\frac{\partial \phi}{\partial x} + \mathrm{v}\frac{\partial \phi}{\partial y}\right)\right] + \frac{\partial N_i}{\partial y}\left[D\frac{\partial \phi}{\partial y} + \frac{\alpha u \bar{h}}{2|\mathbf{V}|}\left(u\frac{\partial \phi}{\partial x} + \mathrm{v}\frac{\partial \phi}{\partial y}\right)\right] + N_i\left(u\frac{\partial \phi}{\partial x} + \mathrm{v}\frac{\partial \phi}{\partial y}\right)\right\} d\Omega - \int_\Gamma N_i D\frac{\partial \phi}{\partial \mathbf{n}} = 0$$

15

Finally, after some more algebraic manipulations we can write

$$\int_\Omega \left\{D\left(\frac{\partial N_i}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial \phi}{\partial y}\right) + u\left[N_i + \frac{\alpha \bar{h}}{2|\mathbf{V}|}\left(u\frac{\partial N_i}{\partial x} + \mathrm{v}\frac{\partial N_i}{\partial y}\right)\right]\frac{\partial \phi}{\partial x} + \mathrm{v}\left[N_i + \frac{\alpha \bar{h}}{2|\mathbf{V}|}\left(u\frac{\partial N_i}{\partial x} + \mathrm{v}\frac{\partial N_i}{\partial y}\right)\right]\frac{\partial \phi}{\partial y}\right\} d\Omega - \int_\Gamma N_i D\frac{\partial \phi}{\partial \mathbf{n}} = 0$$

This last expression suggests the Petrov – Galerkin weights

$$w_i = N_i + \frac{\alpha \bar{h}}{2|\mathbf{V}|}\left(u\frac{\partial N_i}{\partial x} + \mathrm{v}\frac{\partial N_i}{\partial y}\right)$$

which are a natural extension of the one-dimensional weights, and do not introduce CROSS–FLOW NUMERICAL DIFFUSION

Notice that the contribution to the line integral was omitted, the Functions act only in the interior of the domain. It can be shown That this is the consistent way to formulate the method.

16

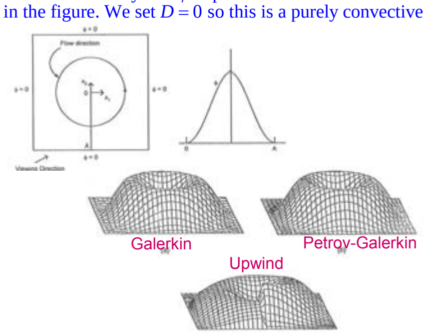
EXAMPLE

Advection of a cosine hill in a rotating field defined by

$u = -y \quad \mathrm{v} = x$

The outside boundary is $|x| = 2$ and $|y| = 2$ where ϕ=0.

Along the internal boundary $\overline{OA}$ ϕ is prescribed as a cosine as shown in the figure. We set $D = 0$ so this is a purely convective situation.



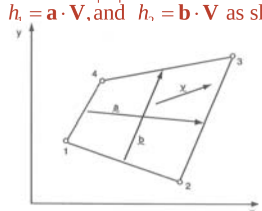
17

In practice the Petrov – Galerkin weights are used only in the convective term and the source term. In 2- and 3- dimensions it amounts to ignoring the cross derivative constants arising from the diffusion term. However it can be shown that this does not affect the accuracy.

The element length $\bar{h}$ in the direction of flow is given by

$\bar{h} = \frac{1}{|\mathbf{V}|}\left(|h_1| + |h_2|\right)$ where h_1 and h_2 are given by

$h_1 = \mathbf{a}\cdot\mathbf{V}$, and $h_2 = \mathbf{b}\cdot\mathbf{V}$ as shown in the figure.



$$a_x = \frac{1}{2}(x_2 + x_3 - x_1 - x_4)$$

$$a_y = \frac{1}{2}(y_2 + y_3 - y_1 - y_4)$$

$$b_x = \frac{1}{2}(x_3 + x_4 - x_1 - x_2)$$

$$b_y = \frac{1}{2}(y_3 + y_4 - y_1 - y_2)$$

18

Algorithms for convective flow stabilization have also been developed for triangles (Tabata, Kikuchi etc.) we will not describe them here.

A significant number of algorithms similar to our Petrov – Galerkin and variations on it have also been proposed. Most notably the so Called *discontinuity capturing* methods. Most of these methods Introduce parameters for which we do not have a clear criterion to Choose, or make the problem non-linear.

The Petrov – Galerkin method can be considered a particular case Of a more general family of algorithms known as the ***Galerkin-Least-Squares method.*** Which we now explain in one dimension.

Write the residual of the one-dimensional convective-diffusion equation $R=-\frac{d}{dx}\left(k\frac{d\phi}{dx}\right)+u\frac{d\phi}{dx}-S=0$ a least squares solution requires the minimization of the functional $I=\int_\Omega R^2 d\Omega$ over the space of trial functions.

Approximating ϕ with shape functions $\phi = N_j\phi_j$ (where repeted indices imply summation) the conditions for the minimum are

$$\frac{\partial}{\partial\phi_i}\int_0^L\left[\left(-\frac{d}{dx}\left(k\frac{dN_j}{dx}\right)+u\frac{dN_j}{dx}\right)\phi_j-S\right]^2 dx=0$$

and this leads to the finite element equations

$$\int_0^L\left[u\frac{dN_i}{dx}-\frac{d}{dx}\left(k\frac{dN_i}{dx}\right)\right]\left[\left(-\frac{d}{dx}\left(k\frac{dN_j}{dx}\right)+u\frac{dN_j}{dx}\right)\phi_j-S\right]dx=0$$

on the other hand, the Galerkin formulation of the equation is

$$\int_0^L N_i\left[\left(-\frac{d}{dx}\left(k\frac{dN_j}{dx}\right)+u\frac{dN_j}{dx}\right)\phi_j-S\right]dx=0$$

The Galerkin – Least- Squares formulation consists in satisfying a linear combination of the two equations above. That is

$$\int_0^L\left[N_i+\tau\left\{u\frac{dN_i}{dx}-\frac{d}{dx}\left(k\frac{dN_i}{dx}\right)\right\}\right]\left[\left(-\frac{d}{dx}\left(k\frac{dN_j}{dx}\right)+u\frac{dN_j}{dx}\right)\phi_j-S\right]dx=0$$

This is clearly a Petrov-Galerkin formulation using the weight function

$$\mathrm{w}_i=N_i+\tau\left(-\frac{d}{dx}\left(k\frac{dN_i}{dx}\right)+u\frac{dN_i}{dx}\right)$$

Using linear elements and defining $\tau=\frac{\alpha h}{2u}$ we recover the original Petrov-Galerkin formulation.

Even more general formulations, the *Generalized-Galerkin-Least-Squares* have been proposed to deal with time dependent Convection-Diffusion problems. Where two-parameters must be determined.

Methods have been proposed in which up to four parameters are added in the formulation.

THREE DIMENSIONS

The extension to three dimensions is obtained by means of a Local rotation in the direction of flow and the addition of an Anisotropic diffusion in the same way as two dimensions. The Petrov - Galerkin weighting functions obtained are

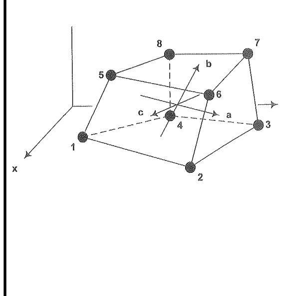
$$w_i=N_i+\frac{\alpha\bar{h}}{2|\mathbf{V}|}\left(u\frac{\partial N_i}{\partial x}+\mathrm{v}\frac{\partial N_i}{\partial y}+w\frac{\partial N_i}{\partial z}\right),\quad \mathbf{V}=\begin{bmatrix}u\\ \mathrm{v}\\ w\end{bmatrix}$$

The parameters α and γ as in 2-D and the length $\bar{h}$ is given by

$$\bar{h}=\frac{1}{|\mathbf{V}|}\left(|h_1|+|h_2|+|h_3|\right)$$

$$h_1=\mathbf{a}\cdot\mathbf{V}$$
$$h_2=\mathbf{b}\cdot\mathbf{V}$$
$$h_3=\mathbf{c}\cdot\mathbf{V}$$

$\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are given by



$$a_x=\frac{1}{4}(x_2+x_3+x_6+x_7-x_1-x_4-x_5-x_8)$$
$$a_y=\frac{1}{4}(y_2+y_3+y_6+y_7-y_1-y_4-y_5-y_8)$$
$$a_z=\frac{1}{4}(z_2+z_3+z_6+z_7-z_1-z_4-z_5-z_8)$$
$$b_x=\frac{1}{4}(x_5+x_6+x_7+x_8-x_1-x_2-x_3-x_4)$$
$$b_y=\frac{1}{4}(y_5+y_6+y_7+y_8-y_1-y_2-y_3-y_4)$$
$$b_z=\frac{1}{4}(z_5+z_6+z_7+z_8-z_1-z_2-z_3-z_4)$$
$$c_x=\frac{1}{4}(x_1+x_2+x_5+x_6-x_3-x_4-x_7-x_8)$$
$$c_y=\frac{1}{4}(y_1+y_2+y_5+y_6-y_3-y_4-y_7-y_8)$$
$$c_z=\frac{1}{4}(z_1+z_2+z_5+z_6-z_3-z_4-z_7-z_8)$$

Non-linear Equations

In simulations of fluid flow we must solve the Burgers equation in One dimension and the Navier-Stokes equations in two and three Dimensions, which are all non-linear in the convective terms.

First consider the Burgers equation $-\varepsilon\frac{d^2u}{dx^2}+u\frac{du}{dx}=0$

1) It was established (H-Z 1979) that using the Petrov-Galerkin method in conjunction with a Newton-Raphson iteration leads to very slow convergence and sometimes instability with strong convection.

2) Using a direct iteration convergence was faster for the convection dominated cases. An alternative was also developed using modified quadrature formulae (Hu.-1979) to resolve these problems.

Example

$$-\varepsilon\frac{d^2u}{dx^2}+u\frac{du}{dx}=0,\; u(0)=1,\; u(1)=0,\quad u^*(x)=A\left(\frac{e^{\frac{A}{\varepsilon}x}-B}{e^{\frac{A}{\varepsilon}x}+B}\right)$$

A and B are integration constants. We will set ε=0.1 and use 10 linear elements using Galerkin and Petrov-Galerkin combined with Newton-Raphson and a direct iteration method. Convergence is determined when the difference at every node between two consecutive iterations is less than 10^{-4}.

Table 8.2 Results for one-dimensional Burgers equation

	Direct Iteration		Newton-Raphson		Analytical
x	Galerkin	Petrov-Galerkin	Galerkin	Petrov-Galerkin	Solution
.2	1.000	0.999	1.000	0.999	0.999
.4	0.997	0.995	0.997	0.995	0.995
.6	0.969	0.963	0.970	0.964	0.964
.8	0.766	0.759	0.770	0.762	0.762
.9	0.460	0.459	0.464	0.462	0.462

Note that the Galerkin solution is non-oscillatory, even though The cell Peclet number *Pe* = 10 in some elements. However, near the boundary layer at x = 1, *Pe* = 0.231.

An important difference with linear problems is that *Pe* is variable and depends on the solution. Therefore it is not possible to predict the onset of oscillations a priori.

25

Table 8.3: Number of iterations needed for convergence.

	Direct iteration		Newton-Raphson	
ε^{-1}	Galerkin	Petrov-Galerkin	Galerkin	Petrov-Galerkin
5	5	5	4	4
10	6	6	5	5
10^2	6*	4	6*	6
10^3	**	4	16*	7
10^6	**	4	**	7

* Solution is oscillatory
** Solution does not converge after 20 iterations

Notice that the Newton-Raphson-Petrov-Galerkin method becomes less efficient as convection dominates.
In simulations of the Navier-Stokes equations eventually it fails to converge (Hu 1979, H-Z 1979).

26

For the Navier-Stokes equations, even though Newton-Raphson Is the method of choice to get steady-state solutions directly, There are three problems that arise:

1) For very highly convective flows the Petrov-Galerkin method must be further modified to achieve convergence. There is no agreement as to what the best way is.
2) For large Reynolds numbers the solution must be obtained incrementally starting from a low Re, then increasing the Re number and using the previous solution as initial guess until the desired value of Re is reached.
3) In many flow problems as Re increases there are bifurcations points where the solution changes to a different mode that is physically more stable. Direct solutions are rarely capable to Switch to the new mode and continue along the branch that is Physically unstable.

For these reasons we will prefer to use time dependent algorithms To reach steady state. These always follow the physically stable Branches and allow us to use Petrov-Galerkin as in linear systems.

27

EXAMPLE

Let us use direct substitution to solve the Burgers equation.

u_i^{n+1} are obtained from $\int_0^1 \left(\varepsilon \frac{dw^k}{dx}\frac{du^{k+1}}{dx} + w^k u^k \frac{du^{k+1}}{dx} \right) dx = 0$

This differs from Galerkin in that the weighting functions $w_i^k = N_i + \frac{\alpha^k h}{2}\frac{dN_i}{dx}$ change at each iteration because the cell Peclet number is $\gamma^k = \frac{\bar{u}^k h}{\varepsilon}$, and $\alpha^k = \coth\left(\frac{\gamma^k}{2}\right) - \frac{1}{\gamma^k}$

We have chosen to apply the Petrov-Galerkin method to the non-linear problems finding α in the same way as in the linear case. It is far from obvious that this should be the right way to do it. However, it was proved (H-E 1982) using Taylor series expansions that at least the first two terms in the error expansion of the Burgers equation are identical to the linear case.

28

EXAMPLE

For the Newton-Raphson iteration the Petrov-Galerkin iteration takes the form $\int_0^1 \left(\varepsilon \frac{dw^k}{dx}\frac{d\Delta u}{dx} + w^k u^k \frac{d\Delta u}{dx} + w^k \frac{du^k}{dx}\Delta u \right) dx = 0$

This equation is of the Convection-Diffusion-Reaction type, and That is the reason why the Petrov-Galerkin method as we have Applied is not sufficient to stabilize the calculations when it is Highly convection dominated.

Various modifications to resolve this situation have been proposed (see e. g. Id 1996). A simple way to improve the stability is to use The Petrov-Galerkin weights in the middle term only, and the shape Functions in the other terms. This is effective to stabilize the Calculations but it sacrifices some of the accuracy.

29