TIME DEPENDENT CONVECTION-DIFFUSION

We now consider the equation $\frac{\partial \phi}{\partial t} + u\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(D\frac{\partial \phi}{\partial x}\right) = S$, with initial condition $\phi(x,0) = f(x)$ and appropriate boundary conditions

Two new numerical difficulties arise that were not present before. These are *NUMERICAL DAMPING* and *NUMERICAL PHASE LAG* and we will illustrate them through the following example:

Let us solve the advection equation $\frac{\partial \phi}{\partial t} + 0.25\frac{\partial u}{\partial x} = 0$ in 0<x<2

with initial condition $\phi(x,0) = e^{-800(x-0.25)^2}$

and boundary conditions $\phi(0,t) = \phi(2,t) = 0.$

We will apply the Crank-Nicolson-Galerkin ($\theta = 1/2$) method with 80 Linear elements, that is $\Delta x = 0.025$, and time step $\Delta t = 0.09$. The initial condition and the solution at $t = 2.07$ and $t = 4.05$ are shown in the figure.

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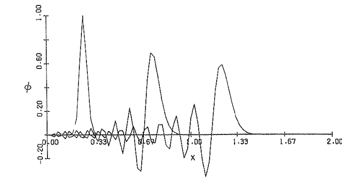


The solution is a pure translation of the initial spike. The analytical expression is $\phi(x,t) = e^{-800\left[x-0.25(t+1)\right]^2}$

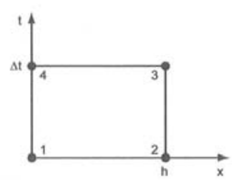
The results are very discouraging:

1) The original amplitude has decreased by about 40% (Damping)
2) The spike that should be at *x* = 0.768 at *t* = 2.07 and at *x* = 1.26 at *t* = 4.05 is only at *x* = 0.73 and *x* = 1.22 respectively (out of phase)
3) There is an enormous amount of numerical "*DISPERSION*" or "*NOISE*" in the solution.

2

Space time elements

Let us now solve $\frac{\partial \phi}{\partial t} + 0.25\frac{\partial u}{\partial x} - 0.0003125\frac{\partial^2 \phi}{\partial x^2} = 0.0$ using the Galerkin method and Linear 2-dimensional space-time elements.



$$N_1(x,t) = \left(1-\frac{x}{h}\right)\left(1-\frac{t}{\Delta t}\right)$$
$$N_1(x,t) = \frac{x}{h}\left(1-\frac{t}{\Delta t}\right)$$
$$N_1(x,t) = \frac{x}{h}\frac{t}{\Delta t}$$
$$N_1(x,t) = \left(1-\frac{x}{h}\right)\frac{t}{\Delta t}$$

The initial condition is chosen so that the exact solution is

$$\phi(x,t) = \frac{1}{\sqrt{1+t}}e^{-\frac{\left[x-0.25(t+1)\right]^2}{\left[4D(t+1)\right]}}$$

The mesh size is $h = 0.025$ and the time step $\Delta t = 0.08$. (this gives $\gamma = 20$)

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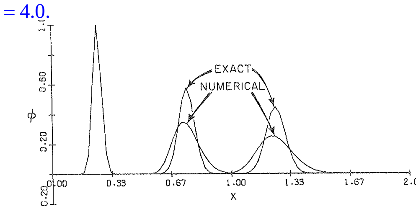
The exact and numerical solution are shown at times $t = 0.0$, $t = 2.0$ and $t = 4.0$.



The amplitude of the solution at $t = 2.0$ is only 60% of the exact value. At $t = 4.0$ it is 55%.
Some phase lag can also be observed.

The space-time finite element scheme used here is equivalent to the θ-method with $\theta = 2/3$.

NUMERICAL DAMPING

Go back to the purely convective equation $\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0$ on $0 < x < 2L$ and write the solution in complex Fourier series form.

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$$\phi = \sum_{n=-\infty}^{\infty}\Phi_n = \sum_{n=-\infty}^{\infty}c_n e^{i\alpha_n(x-at)}$$

The coefficients c_n and the wave numbers α_n depend on he initial condition. Define the *Analytical Amplification Factor* (AAF) as the ratio of the amplitude of the n-th Fourier component at time $t+\Delta t$ to the amplitude at time t. $AAF = \left|\frac{\Phi_n(x,t+\Delta t)}{\Phi_n(x,t)}\right|$. The amplitude increses, decreases or remains the same depending on whether AAF is geater than, less than or equal to 1.

We can consider one component $\Phi_n = c_n e^{i\alpha_n[x-a(t+\Delta t)]}$, then the AAF reduces to $AAF = \left|\frac{\Phi_n(x,t+\Delta t)}{\Phi_n(x,t)}\right| = \left|e^{i\alpha_n\Delta t}\right| = 1$

In the absence of numerical error, the solution must not change the amplitude of the Fourier components in time.

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To understand the behavior of the numerical solution, we write it as a discrete Fourier series $\phi(x_j,t_n) = \sum_{k=-K}^{K}a_k(t)\xi^n e^{i[k_k(j\Delta x)]}$.

The coefficients a_k depend on the initial condition; ξ is a constant; K is the number of elements contained in $0 < x < L$ (one half of the domain); $i = \sqrt{-1}$ and $k_k = \frac{2\pi k}{2L}$ is a wave number.

The corresponding frequencies are $f_k = \frac{k_k}{2\pi}$ and measure the number of wavelengths contained in each $2L$ interval. The lowest, k = 0, corresponds to a time independent term. The highest , $j = K$ has wave number $\frac{2\pi K}{2L} = \pi\left(\frac{K}{L}\right) = \frac{\pi}{\Delta x}$ and gives the number of points that will represent represent a sine wave between 0 and 2π.

The finite series above is the exact solution generated by the numerical method when the a_k are the Fourier coefficients of the initial condition.

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The *Numerical Amplification Factor* is defined as

$$NAF = \left| \frac{\phi_j^{n+1}}{\phi_j^n} \right| = |\xi|$$

The *DAMPING* is the numerical error γ_d in the amplification factor for a component j. $\gamma_d = \frac{NAF}{AAF} = \frac{|\xi|}{1} = |\xi|.$

To propagate a solution without amplification error we need $|\xi| = 1.$

EXAMPLE

Discretize the convective equation $\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0$ using the Crank-Nicolson-Galerkin method with linear elements. We obtain

$$\left(\frac{h}{6\Delta t}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{u}{4}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right)\begin{bmatrix} \phi_1^{n+1} \\ \phi_2^{n+1} \end{bmatrix} = \left(\frac{h}{6\Delta t}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \frac{u}{4}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right)\begin{bmatrix} \phi_1^{n} \\ \phi_2^{n} \end{bmatrix}$$

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Assembling two elements the difference equation for node *j* is

$$\frac{2}{3}\left[\left(\phi_{j-1}^{n+1} + 4\phi_j^{n+1} + \phi_{j+1}^{n+1} \right) - \left(\phi_{j-1}^{n} + 4\phi_j^{n} + \phi_{j+1}^{n} \right) \right] + \frac{u\Delta t}{h}\left[\left(\phi_{j+1}^{n+1} - \phi_{j-1}^{n+1} \right) + \left(\phi_{j+1}^{n} - \phi_{j-1}^{n} \right) \right] = 0$$

Replace $\phi_j^n = \xi^n e^{i(k\cdot jh)}$

$$\frac{2}{3}\left[\xi^{n+1}\left(e^{ik(j-1)h} + 4e^{ikjh} + e^{ik(j+1)h} \right) - \xi^n\left(e^{ik(j-1)h} + 4e^{ikjh} + e^{ik(j+1)h} \right) \right] + \frac{u\Delta t}{h}\left[\xi^{n+1}\left(e^{ik(j+1)h} - e^{ik(j-1)h} \right) + \xi^n\left(e^{ik(j+1)h} - e^{ik(j-1)h} \right) \right] = 0$$

Multiply by ξ^{-n}, $\frac{3}{2}$, and $e^{-ik(j-1)h}$

$$(\xi - 1)\left(1 + 4e^{ikh} + e^{i2kh} \right) + \frac{3u\Delta t}{2h}(\xi + 1)\left(e^{i2kh} - 1 \right) = 0$$

Because *kh* can attain any value we set $\theta = kh.$ Using Euler's equation we have $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{2i\theta} = \left(\cos\theta + i\sin\theta \right)^2$

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$$(\xi - 1)\left(1 + 4(\cos\theta + i\sin\theta) + \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta \right) + \frac{3u\Delta t}{2h}(\xi + 1)\left((\cos\theta + i\sin\theta)^2 - 1 \right) = 0$$

Further manipulations yield

$$\xi\left(\cos\theta(4 + 2\cos\theta) + i\sin\theta(4 + 2\cos\theta) \right) - \left(\cos\theta(4 + 2\cos\theta) + i\sin\theta(4 + 2\cos\theta) \right) + i\frac{3u\Delta t}{h}\xi\sin\theta(\cos\theta + i\sin\theta) + i\frac{3u\Delta t}{h}\sin\theta(\cos\theta + i\sin\theta) = 0$$

And finally

$$\xi = \frac{(4 + 2\cos\theta) - i\frac{3u\Delta t}{h}\sin\theta}{(4 + 2\cos\theta) + i\frac{3u\Delta t}{h}\sin\theta}$$

REMARKS:

1) Noting that k is a wave number, $k = 2\pi / L$ if L is the wave length, then $\theta = kh = k\Delta x = 2\pi\left(\frac{\Delta x}{L} \right)$. This is convenient for us to plot the magnitude of ξ versus the number of elements per wave length in a covenient way.

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2) The expression for ξ is of the form $\frac{z}{\bar{z}}$ therefore $|\xi| \equiv 1$ and the Crank-Nicolson-Galerkin method HAS NO DAMPING. This is good because the algorithm preserves the amplitude of the Fourier components, but it is bad because the method has no mechanism to eliminate perturbations. Moreover

3) We define the Courant number c as $c = \frac{u\Delta t}{h}.$

If the mesh and time step are chosen so that $c = 1$, a particle of fluid will travel exactly the distance *h* in one time step. The peak translates from one node to the next and the solution is excellent.

However, our introductory example has $c = 0.9$, and the results were disastrous. Therefore, the mesh could not capture the peak except every 10 time steps. The algorithm does one of two things:

i) Assume that the peak is attained at the node closest to it, thus changing the speed of propagation and introducing phase lag.

ii) Give the correct value at the closest node, and to conserve mass, redistribute the excess mass throughout the domain. Due to lack of memory (the method only knows what happens at $t = t_n$, but once the amplitude decreases it cannot increase back up. Thus producing damping and noise.

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What actually happens is a combination of the two. This introduces dispersion and numerical damping. Moreover, because there is no damping in the algorithm (|ξ| = 1) there is no mechanism to kill the oscillations so they just stay there and as the amplitude continues to decay they eventually destroy the solution.

EXAMPLE

Apply the Petrov- Galerkin method to the purely convective equation $\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0,$ use the θ-method with $\theta = 1$. The difference equation is

$$\phi_j^{n+1} + c\left(\phi_j^{n+1} - \phi_{j-1}^{n+1} \right) = \phi_j^n$$

and replacing $\phi_j^n = \xi^n e^{ikjh}$ as before we obtain

$$\xi = \frac{1}{1 + c(1 - \cos\theta + i\sin\theta)}$$

from here we get

$$|\xi| = \left[(1 + c)^2 + c^2 - 2c(1 + c)\cos\theta \right]^{-1/2}$$

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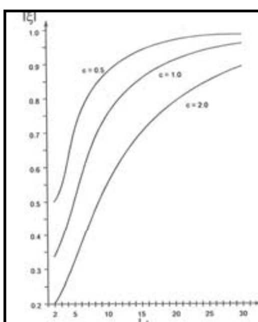

REMARKS:

1) The algorithm has too much damping. For $c = 1$, $|\xi| = 0.577$ if six elements per wavelength are used, and $|\xi| = 0.862$ using 15 elements per wave length. 15 elements is very fine resolution and still reduces the amplitude by 14% in one time step.

2) A reasonably accurate scheme should resolve a wave with about 6 elements reasonably well to be affordable.

3) The damping is reduced as the Courant number decreases. However, a lower Courant number requires a smaller time step. Note that for c=2.0 the damping is very strong. In general we should not calculate with a Courant number greater than 1 if we need to capture the time response accurately. But if only the steady-state is of interest the damping provides additional stability.

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4) In this example $\theta = 1$, so the method is unconditionally stable. Because the NAF $|\xi|$ is less than 1, perturbations in the solution will decrease and eventually disappear in time. But if $|\xi|>1$ perturbations will grow and and the algorithm become unstable. The stability is governed by

$$|\xi| \begin{cases} <1 & stable \\ =1 & neutrally\ stable \\ >1 & unstable \end{cases}$$

This criterion for stability is only useful for linear equations, but it can be extended to systems of linear equations as well.

EXAMPLE

Let us now use $\theta = 0$ to discretize the convective equation. The difference equation is $\phi_j^{n+1} = c(\phi_{j-1}^n - \phi_j^n) + \phi_j^n$ fully explicit.

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$|\xi|$ is found to be $|\xi| = \left(1 - 4c(1-c)\sin^2(\theta/2)\right)^{\frac{1}{2}}$
and the stability condition $|\xi| \le 1$ becomes

$$-1 \le 1 - 4c(1-c)\sin^2(\theta/2) \le 1$$

Manipulating this expression algebraically we find that the first inequality is always satisfied.

The second inequality turns out to be satisfied only if $c \le 1$. So the method is stable when $\Delta t \le \frac{h}{u}$

von Neumann's method can be extended to two dimensions by expressing the Fourier components of the numerical solution in the form $\phi\left(x_j, y_\ell, t_n\right) \equiv \phi_{j\ell}^n = \xi^n e^{ik(j\Delta x)} e^{im(\ell\Delta y)}$. The damping characteristic and time step limitations if any can be obtained.

NUMERICAL PHASE ERROR

The phase error is related to the translational velocity of each of the Fourier components of the numerical solution.

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The phase angle of a Fourier component is given by

$$\tau = \tan^{-1}\left(\frac{\mathrm{Im}(\xi)}{\mathrm{Re}(\xi)}\right)$$

In one time step, the real wave moves a distance τ_0 given by $\tau_0 = 2\pi/N$ where N is the number of time steps required to move one full wave length. Therefore, $N = \frac{L}{u\Delta t}$. Replacing in the expression for τ_0 we get $\tau_0 = c\theta.$

The phase error Θ is defined as

$$\Theta = \tau_0 - \tau = c\theta - \tan^{-1}\left(\frac{\mathrm{Im}(\xi)}{\mathrm{Re}(\xi)}\right)$$

EXAMPLE

The phase angle for the explicit convective algorithm $\phi_j^{n+1} = c(\phi_{j-1}^n - \phi_j^n) + \phi_j^n$ is given by $\tau = \tan^{-1}\left(\frac{-c\sin\theta}{1 - c(1-\cos\theta)}\right)$. The phase error Θ is shown in the figure.

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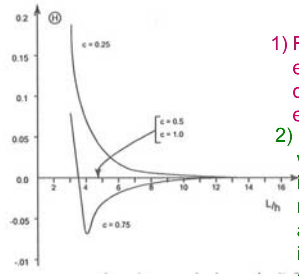

Remarks:

1) For c = 0.25 and 0.5 the phase error can be large. However for c = 0.5 or 1.0 there is no phase error.

2) In a uniform mesh, the shortest wavelength that can be captured is *L = 2h* . However this will give no accuracy. In practice we build algorithms so that the waves of interest can be captured with about 6 elements.

3) When using irregular meshes in hyperbolic problems we must be careful when going from a fine mesh to a coarser one, because the coarser mesh will not be able to capture the smallest wave length carried by the fine mesh. This will result in the internal reflection of the short wave lengths. Modifications to the algorithms are needed to avoid them.

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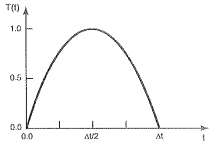
PETROV-GALERKIN METHOD FOR TIME-DEPENDENT CONVECTION-DIFFUSION

There are many ways to obtain stabilized algorithms for time dependent problems. We will only look in detail at a natural extension of the method developed for the steady state case.

We already saw that a method based on bilinear time-space elements is over diffusive. This remains true if Petrov-Galerkin weights are used. To avoid this, we construct weighting functions that are parabolic in time. The time variation is given by

$$T(t) = \frac{4}{\Delta t}\left(1 - \frac{t}{\Delta t}\right)$$



We construct weighting functions of the form $M_i = N_i(x)T(t)$ and number them as usual, counter-clockwise from the lower left node. $N_i(x)$ are the one-dimensional linear shape functions.

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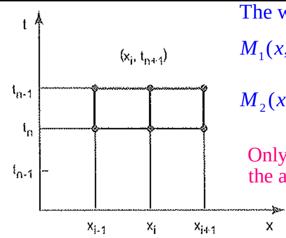


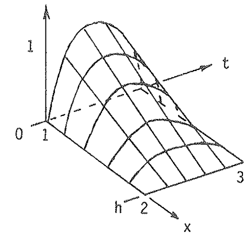
The weighting functions $M_i(x,t)$ are

$$M_1(x,t) = M_4(x,t) = 4\left(1 - \frac{x}{h}\right)\frac{t}{\Delta t}\left(1 - \frac{t}{\Delta t}\right)$$

$$M_2(x,t) = M_3(x,t) = \frac{4xt}{h\Delta t}\left(1 - \frac{t}{\Delta t}\right)$$

Only M_1 and M_2 are needed to define the algorithm.

Function $M_1(x,t)$



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In the steady-state case we added a balancing diffusion, and introduced a parameter α determined from the truncation error analysis.

We now introduce a second term and parameter to balance dispersion. This has the form $\beta d\left(\frac{\partial^3\phi}{\partial x^2\partial t}\right)$, with $d \sim uh\Delta t$ for consistency. The modified equation is

$$\frac{\partial\phi}{\partial t}+u\frac{\partial\phi}{\partial x}-\left(D+\frac{\alpha uh}{2}\right)\frac{\partial^2\phi}{\partial x^2}+\beta d\frac{\partial^3\phi}{\partial x^2\partial t}=0$$

Now we apply a Petrov-Galerkin method using linear space-time shape functions and the functions M_i as weights. Then the weak form is re-arranged as done before for the steady-state case and we obtain the final Petrov-Galerkin weights

$$w_i(x,t)=M_i(x,t)+\frac{\alpha h}{2}\frac{\partial M_i}{\partial x}+\frac{\beta h\Delta t}{4}\frac{\partial^2 M_i}{\partial x\partial t}$$

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The weighting functions become

$$w_1(x,t)=4\left(1-\frac{x}{h}\right)\frac{t}{\Delta t}\left(1-\frac{t}{\Delta t}\right)-2\alpha\frac{t}{\Delta t}\left(1-\frac{t}{\Delta t}\right)-\beta\left(1-\frac{2t}{\Delta t}\right)$$

$$w_2(x,t)=4\frac{x}{h}\frac{t}{\Delta t}\left(1-\frac{t}{\Delta t}\right)+2\alpha\frac{t}{\Delta t}\left(1-\frac{t}{\Delta t}\right)+\beta\left(1-\frac{2t}{\Delta t}\right)$$

$$\int_a^b w\left(\frac{\partial\phi}{\partial t}+u\frac{\partial\phi}{\partial x}-D\frac{\partial^2\phi}{\partial x^2}\right)dx=0$$

The difference equation for node *i* is

$$\frac{1}{9\Delta t}\left[\left(\phi_{i-1}^{n+1}-\phi_{i-1}^{n}\right)+4\left(\phi_{i}^{n+1}-\phi_{i}^{n}\right)+\left(\phi_{i+1}^{n+1}-\phi_{i+1}^{n}\right)\right]$$
$$-\frac{\alpha}{6\Delta t}\left[\left(\phi_{i+1}^{n+1}-\phi_{i+1}^{n}\right)-\left(\phi_{i-1}^{n+1}-\phi_{i-1}^{n}\right)\right]+\frac{u}{6h}\left[\left(\phi_{i+1}^{n+1}+\phi_{i+1}^{n}\right)-\left(\phi_{i-1}^{n+1}+\phi_{i-1}^{n}\right)\right]$$
$$-\left[\frac{u}{6h}(\alpha-\beta)+\frac{D}{3h^2}\right]\left(\phi_{i+1}^{n+1}-2\phi_{i}^{n+1}+\phi_{i-1}^{n+1}\right)$$
$$-\left[\frac{u}{6h}(\alpha+\beta)+\frac{D}{3h^2}\right]\left(\phi_{i+1}^{n}-2\phi_{i}^{n}+\phi_{i-1}^{n}\right)=0$$

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The truncation error is manipulated so that it is written in terms of coefficients of the derivatives $\phi_{xx}, \phi_{tx}, \phi_{txx}, \phi_{txxx}, \ldots$ Here $\phi_{txx}=\frac{\partial^3\phi}{\partial t\partial x^2}$ etc.

The truncation error is

$$TE=\left(\frac{2D}{3\gamma}\right)(\sinh\gamma)\left[1-\left(\alpha+\frac{2}{\gamma}\right)\tanh\left(\frac{\gamma}{2}\right)\right]\left(\phi_i^{n+1}\right)_{xx}$$
$$+\left(\frac{2h}{3\gamma^2}\right)\left[\sinh\gamma-\left(\alpha+\frac{2}{\gamma}\right)(\cosh\gamma-1)\right]\left(\phi_i^{n+1}\right)_{tx}$$
$$+\left(\frac{h^2}{3}\right)\left[\frac{\alpha}{\gamma}+\frac{\beta c}{2}+\frac{2}{\gamma^3}\sinh\gamma\left(1-\tanh\left(\frac{\gamma}{2}\right)\right)-\frac{4}{\gamma^4}(\cosh\gamma-1)\right]\left(\phi_i^{n+1}\right)_{txx}$$
$$+\left(\frac{h^3}{3}\right)\left[\frac{c^2}{3}\left(\frac{1}{\gamma}-\frac{c-\alpha}{4}\right)-\frac{\alpha}{2}\left(\frac{1}{6}-\frac{c}{\gamma}-\frac{2}{\gamma^2}\right)-\frac{1}{6\gamma}+\frac{\beta c^2}{4}\right.$$
$$\left.+\frac{2}{\gamma^4}\sinh\gamma\left(1-\tanh\left(\frac{\gamma}{2}\right)\right)-\frac{4}{\gamma^5}(\cosh\gamma-1)\right]\left(\phi_i^{n+1}\right)_{txxx}+HOT$$

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Notice that β does not appear in the first two terms. Moreover choosing $\alpha=\coth\left(\frac{\gamma}{2}\right)-\frac{2}{\gamma}$ the same as before in the steady state case, the first two terms vanish and the error reduces to

$$TE=\left(\frac{h^2}{3}\right)\left[\frac{\alpha}{\gamma}+\frac{\beta c}{2}-\frac{c^2}{6}\right]\left(\phi_i^{n+1}\right)_{txx}$$
$$+\left(\frac{h^3}{3}\right)\left[\frac{c^2}{3\gamma}-\frac{c^3}{12}-\frac{1}{6\gamma}-\frac{\alpha}{12}+\frac{\alpha c}{2\gamma}+\frac{\alpha c^3}{12}+\frac{\alpha}{\gamma^2}+\frac{\beta c^2}{4}\right]\left(\phi_i^{n+1}\right)_{txxx}+HOT$$

Choosing β to eliminate ϕ_{txx} we have $\beta=\frac{c}{3}-\frac{2\alpha}{\gamma c}$.

Notice that α is only a function of γ, while β depends on γ and α. β as a function of α is shown below for several values of c.

The resulting algorithm is third order in space and second order in time. We will show through examples that it has excellent amplitude and phase conservation properties.

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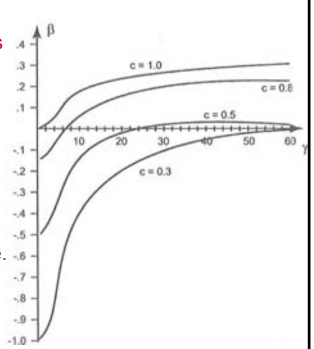
REMARKS:

1) If β = 0 the algorithm reduces to the Petrov- Galerkin scheme for the steady-state equation, and is only second order accurate in space.

2) We can choose the Courant number so that the next term in the error vanishes and get a 4th order algorithm in space. However this is not practical.



3) The algorithm reduces to the Petrov-Galerkin developed before as the solution approaches steady-state. So it is a fully consistent extension.

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4) As $\gamma\to 0$, β becomes undefined because $\beta=\frac{c}{3}-\frac{2\alpha}{\gamma c}$. Physically we must have $u\to 0$, then the difference equation

$$\frac{1}{9\Delta t}\left[\left(\phi_{i-1}^{n+1}-\phi_{i-1}^{n}\right)+4\left(\phi_{i}^{n+1}-\phi_{i}^{n}\right)+\left(\phi_{i+1}^{n+1}-\phi_{i+1}^{n}\right)\right]$$
$$-\frac{\alpha}{6\Delta t}\left[\left(\phi_{i+1}^{n+1}-\phi_{i+1}^{n}\right)-\left(\phi_{i-1}^{n+1}-\phi_{i-1}^{n}\right)\right]+\frac{u}{6h}\left[\left(\phi_{i+1}^{n+1}+\phi_{i+1}^{n}\right)-\left(\phi_{i-1}^{n+1}+\phi_{i-1}^{n}\right)\right]$$
$$-\left[\frac{u}{6h}(\alpha-\beta)+\frac{D}{3h^2}\right]\left(\phi_{i+1}^{n+1}-2\phi_{i}^{n+1}+\phi_{i-1}^{n+1}\right)$$
$$-\left[\frac{u}{6h}(\alpha+\beta)+\frac{D}{3h^2}\right]\left(\phi_{i+1}^{n}-2\phi_{i}^{n}+\phi_{i-1}^{n}\right)=0$$

becomes independent of β and the algorithm reduces to the Crank-Nicolson-Galerkin method.

5) If $\gamma\to\infty$ then $\alpha\to 1$ and $\beta\to c/3$. This is the purely convective case. The difference equation becomes

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$$\frac{1}{18\Delta t}\left[5\left(\phi_{i-1}^{n+1}-\phi_{i-1}^{n}\right)+8\left(\phi_{i}^{n+1}-\phi_{i}^{n}\right)-\left(\phi_{i+1}^{n+1}-\phi_{i+1}^{n}\right)\right]$$
$$+\frac{u}{3h}\left[\phi_{i}^{n+1}-\phi_{i-1}^{n+1}+\frac{\Delta t}{6h}\left(\phi_{i+1}^{n+1}-2\phi_{i}^{n+1}+\phi_{i-1}^{n+1}\right)\right.$$
$$\left.+\phi_{i}^{n}-\phi_{i-1}^{n}-\frac{\Delta t}{6h}\left(\phi_{i+1}^{n}-2\phi_{i}^{n}+\phi_{i-1}^{n}\right)\right]=0$$

and the truncation error is

$$TE=\frac{h^3}{36}(c^2-1)(\phi_i^{n+1})_{txxx}+HOT$$

This algorithm is 3d order accurate. If c = 1 the leading term vanishes and the method is super convergent. If β = 0 we go back to a second order scheme.

6) Many more Petrov-Galerkin and other methods have been proposed for the transient case. **A *BEST* scheme DOES NOT EXIST.**

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7) Various combinations of space-time weights, lumped and consistent mass matrix and inconsistent weighting have been studied. The main conclusions are:

i) Consistent mass and weighting produces the best accuracy. Mass lumping does improve accuracy when bilinear weights are used in the purely convective case. But for the quadratic in time weighting functions consistent mass is always superior.

ii) The Petrov-Galerkin weights are not unique. Any function of time that is symmetric about *t = Δt/2* and that is not constant can be used to produce similar difference equations.

STABILITY ANALYSIS

The amplification factor of equation (PG) is

$$\xi=\frac{\left[\frac{2}{9}\left(1-2\cos^2(\theta/2)\right)-\frac{2c}{3}\left(\alpha+\beta+\frac{2}{\gamma}\right)\sin^2(\theta/2)\right]-i\left[\left(\frac{c+\alpha}{3}\right)\sin\theta\right]}{\left[\frac{2}{9}\left(1+2\cos^2(\theta/2)\right)+\frac{2c}{3}\left(\alpha-\beta+\frac{2}{\gamma}\right)\sin^2(\theta/2)\right]+i\left[\left(\frac{c-\alpha}{3}\right)\sin\theta\right]}$$

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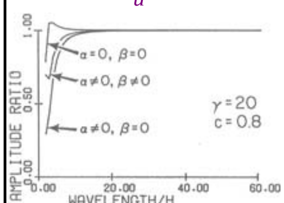
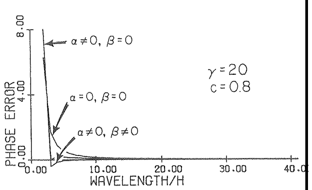
1) If $\beta=0$ the algorithm is unconditionally stable for all α.

2) If $\beta\neq 0$ the method is conditionally stable, $|\xi|\leq 1$ only if $c\leq 1$, that is if $\Delta t\leq \frac{h}{u}$. The case $\gamma=20$, $c=0.8$ is shown below.





We see some damping for small values of *L/h*. However for *L/h*=6 It is only ~3% and ~1% for *L/h*=10

There is some phase error when β=0, but practically no error when $\beta\neq 0$ and L/h ≥ 5.

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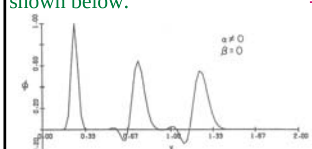
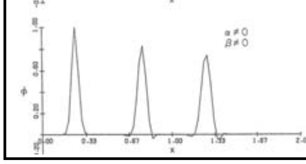
Examples

1) Lets eturn to the advection equation $\frac{\partial\phi}{\partial t}+0.25\frac{\partial\phi}{\partial x}=0$ in 0<x<2 with initial condition $\phi(x,0)=e^{-800(x-0.25)^2}$ and $c=0.9$. The results obtained with $\alpha\neq 0$ and $\beta=0$, and with both $\alpha\neq 0$ and $\beta\neq 0$ are shown below.





The figures show the initial condition and the numerical approximation after the wave has traveled 2 and 4 wave lengths. There must be some damping because c=0.9. If c=1 there is no damping. The max error is shown below.

	t = 2.073	t = 4.055
α = β = 0	0.472	0.520
α ≠ 0 β = 0	0.414	0.465
α ≠ 0 β ≠ 0	0.129	0.198

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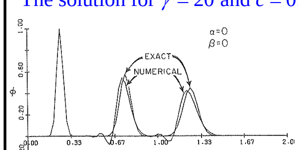
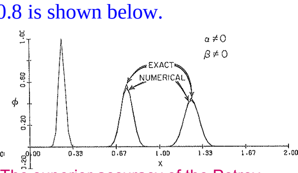
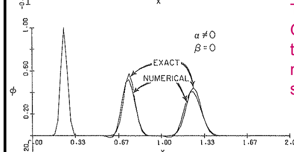
2) Now return to the equation with diffusion

$\frac{\partial\phi}{\partial t}+0.25\frac{\partial\phi}{\partial x}-0.0003125\frac{\partial^2\phi}{\partial x^2}=0.0$ using Petrov-Galerkin.

The solution for $\gamma=20$ and $c=0.8$ is shown below.







The superior accuracy of the Petrov-Galerkin method is evident.Also observe that for α = β = 0 this solution is much more accurate than with bilinear time-space elements. The maximum error is

	t = 2.05	t = 4.05
α = 0, β = 0	0.15	0.11
α ≠ 0, β = 0	0.062	0.052
α ≠ 0, β ≠ 0	0.03	0.028

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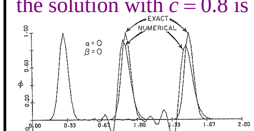
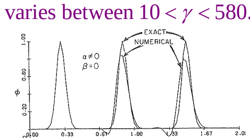
3) We now apply the method to an example with $u=0.3$, and a variable diffusion and source term given by

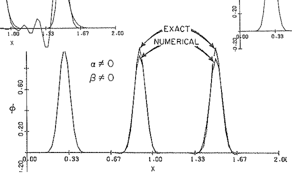
$D(x)=a^2e^{x^2}$

$S(x,t)=2a\left\{1+2x\left[x-u(t+1)\right]-\frac{2}{a}\left[x-u(t+1)^2\right]\right\}e^{\left(x^2-\left[x-u(t+1)\right]^2/a\right)}$

The analytical solution is $\phi(x,t)=e^{-\left[x-u(t+1)\right]^2/a}$. For $a=0.00359$ the solution with $c=0.8$ is shown. γ varies between $10<\gamma<580$.







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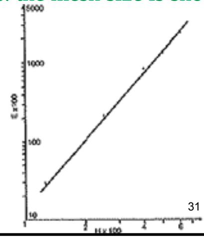
The order of convergence predicted for equation (PG) by the truncation eror analysis when $\alpha \neq 0$ and $\beta \neq 0$ is $O(h^3)$.
To show this, we solve example 2 again, that is

$$\frac{\partial \phi}{\partial t} + 0.25\frac{\partial \phi}{\partial x} - 0.0003125\frac{\partial^2 \phi}{\partial x^2} = 0.0$$

with 5 different meshes but keeping $c = 0.9$ constant. The maximum relative errors are shown in the table below at time $t = 2.07$.
A log-log plot of the relative error vs. the mesh size is shown in the figure.

h	Max absolute error	Max relative error
0.06250	0.149	26.2
0.05000	0.077	14.0
0.04167	0.046	8.7
0.025	0.012	2.2
0.00125	0.002	0.3



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