ME-504: Solution to HW # 5

1. The maximum error occurs at x = 0.5 so we only need to look at that point. Exact solution is 10.7977.

No. of Elements	Δt	θ	$T(0.5, 0.5)$	Error(%)
10	0.005	1.0	10.8173	0.00182
	0.005	0.5	10.4916	0.283
	0.001	1.0	10.5579	0.0222
	0.001	0.5	10.4929	0.028
20	0.005	1.0	11.0482	0.2505
	0.005	0.5	10.7196	0.00723
	0.001	1.0	10.7864	0.00105
	0.001	0.5	10.7209	0.0071
40	0.005	1.0	11.1065	0.0286
	0.005	0.5	10.7771	0.0019
	0.001	1.0	10.5579	0.0043
	0.001	0.5	10.7784	0.00179

2. The matrices **C** and **K** are $\mathbf{C} = \frac{1}{24}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$ and $\mathbf{K} = D_T\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$

The eigenvalue problem is
$$\left| D_T\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} - \frac{\lambda}{24}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \right| = 0$$

The determinant becomes
$$\left(8D_T - \frac{4\lambda}{24}\right)^3 - 2\left(8D_T - \frac{4\lambda}{24}\right)\left(4D_T + \frac{4\lambda}{24}\right) = 0$$

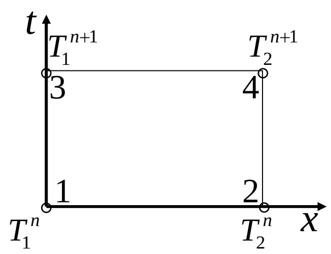
And the eigenvalues are $\lambda_1 = 10.39 D_T$, $\lambda_2 = 48 D_T$, $\lambda_3 = 126.8 D_T$.

The stability limitation is $\Delta t < \frac{2}{\lambda_3} = \frac{0.01577}{D_T}$

3. The shape functions are:

$$N_1(x,t) = \left(1 - \frac{x}{\Delta x}\right)\left(1 - \frac{t}{\Delta t}\right)$$
$$N_1(x,t) = \frac{x}{\Delta x}\left(1 - \frac{t}{\Delta t}\right)$$
$$N_1(x,t) = \frac{x}{\Delta x}\frac{t}{\Delta t}$$
$$N_1(x,t) = \left(1 - \frac{x}{\Delta x}\right)\frac{t}{\Delta t}$$

To find the element stiffness equations for the diffusion equation $\frac{\partial T}{\partial t} = D_T \frac{\partial^2 T}{\partial x^2}$ let us use the following notation:



and since we only need two weighting functions set $W_1(x,t) = \left(1 - \frac{x}{\Delta x}\right)\frac{t}{\Delta t}$

$W_2(x,t) = \frac{x}{\Delta x}\frac{t}{\Delta t}$

The weak form is $\int_0^{\Delta t}\int_0^{\Delta x}\left\{W_i \frac{\partial T}{\partial t} + D_T \frac{\partial W_i}{\partial x}\frac{\partial T}{\partial x}\right\}dx\,dt = 0$ where i = 1, 2 and the temperature is $T(x,t) = N_1(x,t)T_1^n + N_2(x,t)T_2^n + N_4(x,t)T_1^{n+1} + N_3(x,t)T_2^{n+1}$ using the numbering in the expressions given above for the shape functions.

Performing the integrals for W_1 and W_2 we get

$$\frac{\Delta x}{12}[-2T_1^n - T_2^n + 2T_1^{n+1} + T_2^{n+1}] + \frac{D_T \Delta t}{6\Delta x}[T_1^n - T_2^n + 2T_1^{n+1} - 2T_2^{n+1}] = 0$$

$$\frac{\Delta x}{12}[-T_1^n - 2T_2^n + T_1^{n+1} + 2T_2^{n+1}] + \frac{D_T \Delta t}{6\Delta x}[-T_1^n + T_2^n - 2T_1^{n+1} + 2T_2^{n+1}] = 0$$

and manipulating algebraically

$$\left\{\frac{\Delta x}{12}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} + \frac{D_T \Delta t}{3\Delta x}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right\}\begin{bmatrix}T_1^{n+1}\\ T_2^{n+1}\end{bmatrix} = \left\{\frac{\Delta x}{12}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} - \frac{D_T \Delta t}{6\Delta x}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right\}\begin{bmatrix}T_1^n\\ T_2^n\end{bmatrix}$$

which is the $\theta-$ method given in Eq. (7.12) with $\theta = 2/3$.

3. The equation to be solved is $m\ddot{x} + F(x,\dot{x}) - g(t) \equiv G(x,\dot{x},\ddot{x},t) = 0$

Linearize F: $F(x^{n+1},\dot{x}^{n+1}) \simeq F(x^n,\dot{x}^n) + \frac{\partial F(x^n,\dot{x}^n)}{\partial x}(x^{n+1} - x^n) + \frac{\partial F}{\partial \dot{x}}(x^n,\dot{x}^n)(\dot{x}^{n+1} - \dot{x}^n)$

Or $F(x^{n+1},\dot{x}^{n+1}) \simeq F(x^n,\dot{x}^n) + k(x^n,\dot{x}^n)(x^{n+1} - x^n) + c(x^n,\dot{x}^n)(\dot{x}^{n+1} - \dot{x}^n)$

Hence the linearized equation to solve is

$$m\ddot{x}^{n+1} + F(x^n,\dot{x}^n) + k(x^n,\dot{x}^n)(x^{n+1} - x^n) + c(x^n,\dot{x}^n)(\dot{x}^{n+1} - \dot{x}^n) = G(x^{n+1},\dot{x}^{n+1},\ddot{x}^{n+1},t_{n+1}) \equiv G^{n+1}$$

Now set up the iteration by substituting in terms of the predictors

$x^{n+1} = p_{i+1}\,,\ \dot{x}^{n+1} = \dot{p}_{i+1}\,,\ \ddot{x}^{n+1} = \ddot{p}_{i+1}$ and $x^{n} = p_{i}\,,\ \dot{x}^{n} = \dot{p}_{i}\,,\ \ddot{x}^{n} = \ddot{p}_{i}$

$$m\ddot{p}_{i+1} + k(p_i,\dot{p}_i)(p_{i+1}-p_i) + c(p_i,\dot{p}_i)(\dot{p}_{i+1}-\dot{p}_i) = G^{n+1} - F(p_i,\dot{p}_i)$$

Substituting $p_{i+1} = p_i + \Delta p_{i+1},\ \ \dot{p}_{i+1} = \dot{p}_i + \gamma\,\Delta t\,\ddot{p}_{i+1}$ and $\ddot{p}_{i+1} = \dfrac{p_{i+1}-p_0}{\beta\Delta t^2}$ we have

$$m\left(\frac{p_{i+1}-p_0}{\beta\Delta t^2}\right) + k(p_i,\dot{p}_i)\Delta p_{i+1} + c(p_i,\dot{p}_i)\left(\dot{p}_{i+1}-\dot{p}_i\right) = G^{n+1} - F(p_i,\dot{p}_i)\ \ (*)$$

1) $$\frac{p_{i+1}-p_0}{\beta\Delta t^2} = \frac{\Delta p_{i+1}}{\beta\Delta t} + \frac{p_i - p_0}{\beta\Delta t^2} = \frac{\Delta p_{i+1}}{\beta\Delta t} + \ddot{p}_i$$

2) $$\begin{aligned}\dot{p}_{i+1} - \dot{p}_i &= \dot{p}_0 + \gamma\,\Delta t\,\ddot{p}_{i+1} - p_i = \dot{p}_0 + \gamma\,\Delta t\left(\frac{p_{i+1}-p_0}{\beta\Delta t^2}\right) - \dot{p}_i \\ &= \dot{p}_0 + \frac{\gamma}{\beta\,\Delta t}(p_{i+1}-p_0) - \dot{p}_0 - \gamma\,\Delta t\,\ddot{p}_i \\ &= \frac{\gamma}{\beta\,\Delta t}(p_{i+1}-p_0) - \frac{\gamma}{\beta\,\Delta t}(p_i - p_0) \\ &= \frac{\gamma}{\beta\,\Delta t}(p_{i+1}-p_i) = \frac{\gamma}{\beta\,\Delta t}\Delta p_{i+1}\end{aligned}$$

Substitute into (*) :

$$\frac{m}{\beta\Delta t^2}\Delta p_{i+1} + m\,\ddot{p}_i + k(p_i,\dot{p}_i)\Delta p_{i+1} + \frac{\gamma}{\beta\,\Delta t}c(p_i,\dot{p}_i)\Delta p_{i+1} = G^{n+1} - F(p_i,\dot{p}_i)$$

and finally

$$\left[\frac{m}{\beta\Delta t^2} + \frac{\gamma}{\beta\,\Delta t}c(p_i,\dot{p}_i) + k(p_i,\dot{p}_i)\right]\Delta p_{i+1} + +\Delta p_{i+1} = G^{n+1} - F(p_i,\dot{p}_i) - m\,\ddot{p}_i = \Delta Q^{i+1}(p,\dot{p}_i)$$