Homework 6, Due March 11

1. Using ladder operators to calculate the values of $\langle x \rangle$, $\langle x^2 \rangle$, $\langle \hat{p} \rangle$, $\langle \hat{p}^2 \rangle$ for a harmonic oscillator in its ground state and show that $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ and $\Delta p = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2}$ satisfy the uncertainty principle.

2. a) Show that the spherical harmonics are eigenfunctions of the operator $\hat{l}_x^2 + \hat{l}_y^2$ and determine the corresponding eigenvalues.
   b). Calculate the possible angles between the angular momentum vector ($\vec{l}$) and the z-axis, assuming l=1.

3. Prove the following commutating relationships.

$$[\hat{l}_+, \hat{l}_z] = -\hbar \hat{l}_+, \ [\hat{l}_-, \hat{l}_z] = \hbar \hat{l}_-, \text{ and } [\hat{l}_+, \hat{l}_-] = 2\hbar \hat{l}_z .$$

4. In a two-electron system, prove that the following spin wave function is an eigenfunction of both the total spin operator squared ($\hat{S}^2 = (\hat{S}_1 + \hat{S}_2)^2$) and the total projection of $\hat{S}$ on the z-axis ($\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$):

$$|\psi\rangle = |\alpha(1)\rangle |\beta(2)\rangle - |\beta(1)\rangle |\alpha(2)\rangle$$

Determine the corresponding eigenvalues (Hint: make use of the ladder operators similar to those discussed for the regular angular momentum).