Proof of Lemma 1. The result follows since Q is i) nonempty, ii) bounded above, and iii) closed.

i) Choose any $\overline{K} > 0$. If $s(t) = F(\overline{K}, t, \alpha)/2$, for $t = 1,\ldots,T$, then $p(t) > 0$, for all $t = 0,\ldots,T$, and, whereas $y(0) < 0$, $y(t) > 0$, for $t = 1,\ldots,T$. If $r \in (-\infty, \infty)$ is sufficiently small, it follows that $\sum_{t=0}^{T} p(t)y(t)e^{-rt} \geq 0$, so that $r \in Q$.

ii) The assumptions on C and F in the first two paragraphs of 'The Set Up' imply that there exist constants $B > 0$ and $C > 0$ such that $C(K) \geq BK + C$, for all $K \geq 0$, and a function $A(t) > 0$ such that $F(K, t, \alpha) \leq A(t)K$ for all $K \geq 0$, $\alpha \in A$, and $t = 1,\ldots,T$. Consider any $r \in Q$, so that $\sum_{t=0}^{T} p(t)y(t)e^{-rt} \geq 0$, for some $(p, y) = (p(s), y(K, s)) \in M$. Since $p(t) \leq 1$ and $y(t) = F(K, t, \alpha) - s(t) \leq A(t)K$, for $t = 1,\ldots,T$, it follows that $K\left(-B + \sum_{t=1}^{T} A(t)e^{-rt}\right) - C \geq 0$.

Hence $r < \bar{r}$ where $\bar{r}$ is the unique solution of $-B + \sum_{t=1}^{T} A(t)e^{-\bar{r}t} = 0$, and $\bar{r}$ is an upper bound for Q, as required.

iii) Consider any sequence $r_n \to \bar{r}$ where $r_n \in Q$ for all n. It must be shown that $\bar{r} \in Q$. Suppose the sequences K_n and s_n generate r_n. Take N such that $n > N$ implies $r_n \geq \bar{r} - \Delta$, for any fixed $\Delta > 0$. Since $-C(K_n) + \sum_{t=1}^{T} F(K_n, t, \alpha)e^{-(\bar{r}-\Delta)t} \geq 0$, there can be no subsequence of $K_n \to \infty$, given the conditions on C and F. Hence there is a convergent subsequence of $K_n \to \overline{K} \in [0, \infty)$, say. Now define N such that $n > N$ implies $r_n \in (\bar{r} - \Delta, \bar{r} + \Delta)$ and $K_n \in (\overline{K} - \Delta, \overline{K} + \Delta)$, for some $\Delta > 0$. Since $-B(\overline{K} - \Delta) - C + \sum_{t=1}^{T}\left(A(t)(\overline{K} + \Delta)e^{-(\bar{r}-\Delta)t} - s_n(t)e^{-(\bar{r}+\Delta)t}\right) \geq 0$, there can be no subsequence of the $s_n(t) \to \infty$, for any $t = 1,\ldots,T$. Hence there is a convergent subsequence $s_n \to \bar{s} \in [0, \infty)^T$, say. It follows from continuity of the functions $p(s)$ and $y(K,s)$ that $\overline{K}$ and $\bar{s}$ generate $\bar{r}$, so that $\bar{r} \in Q$, as required.

Proof of Lemma 2. Consider backwards induction on t for the results in the first sentence. These results clearly hold at $t = T$. Suppose then, as the induction hypothesis, that they hold at $t + 1$. It follows that $V(K,t,\alpha) \geq 0$ is continuous in $K \geq 0$, since $V(K,t+1,\alpha) \geq 0$ and $F(K,t,\alpha) \geq 0$ are continuous, and $V(K,t,\alpha) = \max_{s(t)}\{F(K,t,\alpha) - s(t) + e^{-r}\sigma(s(t))V(K,t+1,\alpha)\}$. Indeed, the implicit function theorem implies the optimal $s(t)$ satisfying $\sigma'(s(t))V(K,t+1,\alpha)e^{-r} = 1$ is a continuously differentiable function of $K > 0$, for any $\alpha \in A$. Using the 'envelope theorem,' it follows that $V_K(K,t,\alpha) = F_K(K,t,\alpha) + e^{-r}\sigma(s(t))V_K(K,t+1,\alpha)$, $t = 1,\ldots,T - 1$. That is, although $s(t)$ is a function of K, this does not affect this expression because $s(t)$ is chosen optimally. Clearly, $V_K(K,t,\alpha) > 0$ since $F_K(K,t,\alpha) > 0$ and $V_K(K,t+1,\alpha) > 0$. Further, $V_K(K,t,\alpha)$ can be differentiated to yield $V_{KK}(K,t,\alpha)$ as a continuous function of $K > 0$, completing the induction argument.

Consider now choice of K to maximize $\bar{p}V(K,1,\alpha)e^{-r} - C(K)$. The assumptions on F and C imply that $\bar{p}V_K(K,1,\alpha)e^{-r} - C'(K) > 0$, for all small enough $K > 0$, and that $\bar{p}V_K(K,1,\alpha)e^{-r} - C'(K) < 0$ for all large enough K. Hence there must exist an optimal $K > 0$ satisfying the first-order and second-order necessary conditions as stated.

Proof of Proposition 2. The dependence of variables on r is noted. For any $K > 0$, the envelope theorem implies $V_r(K,t,\alpha,r) = -e^{-r}\sigma(s(t))V(K,t+1,\alpha,r) + e^{-r}\sigma(s(t))V_r(K,t+1,\alpha,r)$, $t = 1,\ldots,T - 1$. Since $V_r(K,T,\alpha,r) = 0$, it follows that $V_r(K,t,\alpha,r) < 0$, for $t = 1,\ldots,T - 1$. Given $\bar{p}V(K^*(r^*),1,\alpha,r^*)e^{-r^*} = C(K^*(r^*))$ and $\bar{p}V_K(K^*(r),1,\alpha,r)e^{-r} = C'(K^*(r))$, it follows that $\frac{d}{dr}(\bar{p}V(K^*(r),1,\alpha,r)e^{-r^*} - C(K^*(r))) = \bar{p}V_r(K^*(r),1,\alpha,r) < 0$, so that $\bar{p}V(K^*(r),1,\alpha,r)e^{-r^*} < C(K^*(r))$, for all $r > r^*$. That is, $L(r^*, p^*, y^*) = 0$, so that the growth rate r^* is feasible, but $\max_{p,y} L(r, p, y) < 0$, for all $r > r^*$, so no growth rate strictly greater than r^* is feasible.

Proof of Lemma 3. The optimal K* and s* solve the following problem

$\max_{K,s(1),...,s(T-1)}\left[-C(K)+\sum_{t=1}^{T}\bar{p}\left(\prod_{\tau=1}^{t-1}\sigma(s(t))\right)(F(K,t,\alpha)-s(t))e^{-r^*t}\right]$. The dynamic programming approach in Lemma 2 can be extended to prove that such $K^* > 0$ and $s^* > 0$ are continuously differentiable functions of r^* and α. Since, in addition,

$\left[-C(K^*)+\sum_{t=1}^{T}\bar{p}\left(\prod_{\tau=1}^{t-1}\sigma(s^*(\tau))\right)(F(K^*,t,\alpha)-s^*(t))e^{-r^*t}\right]=0$, the implicit function theorem then implies that the maximum growth rate, $r^*(\alpha)$, say, is a continuously differentiable function of $\alpha\in A$. However, as another example of the envelope theorem, the derivatives of K* and s* play no direct role here. That is, $\dfrac{dr^*(\bar{\alpha})}{d\alpha}=\dfrac{\sum_{t=1}^{T}p^*(t)F_{\alpha}(K^*,1,\bar{\alpha})}{\sum_{t=1}^{T}tp^*(t)\left(F(K^*,t,\bar{\alpha})-s^*(t)\right)}=0$, as required,

given also that $\sum_{t=1}^{T}tp^*(t)\left(F(K^*,t,\bar{\alpha})-s^*(t)\right)=\sum_{t=1}^{T}p^*(t)V(K^*,t,\bar{\alpha})>0$, by Lemma 2.

Proof of Theorem 1. Note that $r^*(\bar{\alpha})=0$ and $\dfrac{dr^*(\bar{\alpha})}{d\alpha}=0$ throughout. Consider first:

Lemma A. (i) $V_{\alpha}(K,t,\bar{\alpha})>0$, for all $K > 0$, and $t=2,...,T$. (ii) $V_{K\alpha}(K,1,\bar{\alpha})>0$, for all $K > 0$.

Proof of Lemma A. (i) By the envelope theorem,
$V_{\alpha}(K,t,\bar{\alpha})=F_{\alpha}(K,t,\bar{\alpha})+\sigma(s(t))V_{\alpha}(K,t+1,\bar{\alpha})$, for all $K>0$, $t=1,...,T-1$. Recall that $F_{\alpha}(K,t,\bar{\alpha})<0$, for all $t<\bar{t}$, but $F_{\alpha}(K,t,\bar{\alpha})>0$, for all $t\geq\bar{t}$. Hence backwards recursion from T implies that $V_{\alpha}(K,t,\bar{\alpha})>0$, for $t=\bar{t},...,T$. Moreover, if $V_{\alpha}(K,t,\bar{\alpha})\leq 0$, for some $t<\bar{t}$, then $V_{\alpha}(K,t-1,\bar{\alpha})<0$. But, since $V_{\alpha}(K,1,\bar{\alpha})=\sum_{t=1}^{T}p^*(t)F_{\alpha}(K,t,\bar{\alpha})/\bar{p}=0$, it must then be that $V_{\alpha}(K,t,\bar{\alpha})>0$, for any $K > 0$, and $t=2,...,T$.

(ii) Differentiating $\sigma'(s(t))V(K,t+1,\alpha)e^{-r^*}=1$ with respect to α, at $\alpha=\bar{\alpha}$, holding

$K > 0$ constant, yields $\frac{\partial s(t)}{\partial \alpha} = -\frac{\sigma'(s(t))V_{\alpha}(K, t+1, \overline{\alpha})}{\sigma''(s(t))V(K, t+1, \overline{\alpha})} > 0$, for $t = 1,\ldots,T-1$. The envelope theorem implies that $V_K(K,1,\alpha) = \sum_{t=1}^{T} e^{-r^*(t-1)}\sigma(s(1))\ldots\sigma(s(t-1))F_K(K,t,\alpha) > 0$, for all $K \geq 0$. Since $F_{K\alpha}(K,t,\alpha) \geq 0$, differentiation of $V_K(K,1,\alpha)$ with respect to α, at $\alpha = \overline{\alpha}$, with $K > 0$ constant, then yields $V_{K\alpha}(K,1,\overline{\alpha}) > 0$.

Now Lemmas 2 and A imply the results of Theorem 1:

(I) Since $\overline{p}V_{KK}(K^*,1,\alpha) - C''(K^*) < 0$, it follows from differentiating $\overline{p}V_K(K^*,1,\alpha)e^{-r^*} = C'(K^*)$ with respect to α, at $\alpha = \overline{\alpha}$, that $\frac{dK^*}{d\alpha} = \frac{\overline{p}V_{K\alpha}(K^*,1,\overline{\alpha})}{C''(K^*) - \overline{p}V_{KK}(K^*,1,\overline{\alpha})} > 0$.

(II) Differentiating $\sigma'(s^*(t))V(K^*, t+1, \alpha)e^{-r^*} = 1$ with respect to α, at $\alpha = \overline{\alpha}$, where K can vary, finally yields $\frac{ds^*(t)}{d\alpha} = -\frac{\sigma'(s^*(t))\left[V_{\alpha}(K^*, t+1, \overline{\alpha}) + V_K(K^*, t+1, \overline{\alpha})\frac{dK^*}{d\alpha}\right]}{\sigma''(s^*(t))V(K^*, t+1, \overline{\alpha})} > 0$, for $t = 1,\ldots,T-1$.