Homework 5 Solutions

Due Nov. 10, 2016

Note

For both the problems, you might want to use the Fourier transform. For the second problem, you might find the method of characteristics useful.

Problem 1 - Solving the Advective Diffusion Equation

Given that the initial probability density $P(x, t = 0)$ of a biased random walker moving on a 1-dimensional continuum is a Gaussian of width w, calculate it at all times t i.e. P(x,t). By taking the moments of $P(x, t)$, i.e. doing the x-integrals, calculate the time evolution of the mean displacement $\langle x(t)\rangle$ and the mean square displacement $\langle x^2(t)\rangle$. Show plots of P(x,t) at the initial and two more times.

The advective diffusion equation combines both diffusion and a travelling wave

$$\frac{\partial P(x,t)}{\partial t} = v\frac{\partial P(x,t)}{\partial x} + D\frac{\partial^2 P(x,t)}{\partial x^2} \tag{1}$$

We can take the Fourier transform of Eq.(1)

$$\begin{aligned}\frac{\partial \widetilde{P}(k,t)}{\partial t} &= ikv\widetilde{P}(k,t) - Dk^2\widetilde{P}(k,t)\\ &= \widetilde{P}(k,t)(-Dk^2 + ikv)\\ \widetilde{P}(k,t) &= e^{-Dk^2+ikv)t}\widetilde{P}(k,0)\end{aligned}$$

The inverse F.T of the above solution gives

$$P(x,t) = \int_{\infty}^{\infty} dk e^{ikx}\left[e^{(-Dk^2+ikv)t}\widetilde{P}(k,0)\right]$$

$$P(x,t) = \int_{\infty}^{\infty} dk e^{-(Dt)k^2+ik(x+vt)}\widetilde{P}(k,0) \tag{2}$$

The initial condition of a Gaussian of width w

$$P(x,0) = \frac{1}{\sqrt{2\pi w^2}}e^{-\frac{x^2}{2w^2}}$$

Then

$$\begin{aligned}\widetilde{P}(k,0) &= \frac{1}{2\pi}\int_{\infty}^{\infty} dx\frac{1}{\sqrt{2\pi w^2}}e^{-\frac{x^2}{2w^2}}e^{-ikx}\\ &= \frac{1}{2\pi}\frac{1}{\sqrt{2\pi w^2}}\sqrt{2w^2}\pi e^{-\frac{k^2w^2}{2}}\\ &= \frac{1}{2\pi}e^{-\frac{k^2w^2}{2}}\end{aligned}$$

Substituting $\widetilde{P}(k,0)$ into Eq.(2)

$$\begin{aligned}
P(x,t) &= \frac{1}{2\pi} = \int_{\infty}^{\infty} dk e^{-(Dt)k^2+ik(x+vt)} e^{-\frac{k^2w^2}{2}} \\
&= \frac{1}{2\pi}\int_{\infty}^{\infty} dk e^{-(Dt+\frac{w^2}{2})k^2+ik(x+vt)} \\
&= \frac{1}{2\pi}\left[\sqrt{\frac{\pi}{Dt+\frac{w^2}{2}}}\exp\left(-\frac{(x+vt)^2}{4(Dt+\frac{w^2}{2})}\right)\right]
\end{aligned}$$

$$\boxed{P(x,t) = \left[\sqrt{\frac{1}{4\pi(Dt+\frac{w^2}{2})}}\exp\left(-\frac{(x+vt)^2}{4(Dt+\frac{w^2}{2})}\right)\right]}$$

$$\begin{aligned}
\langle m\rangle &= \int_{\infty}^{\infty} xP(x,t)dx = -vt \\
\Delta x^2 &= 2Dt + w^2 \\
\langle m^2\rangle &= \int_{\infty}^{\infty} x^2P(x,t)dx = v^2t^2 + 2Dt + w^2
\end{aligned}$$

Problem 2 - Solving the Smoluchowski Equation

Do the exact same problem for a random walker that is pulled towards a point (NOT THE ORIGIN) via a Hookes law interaction.

The Smoluchowski equation is

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\big[\gamma(x-x_0)P(x,t)\big] + D\frac{\partial^2 P(x,t)}{\partial x^2} \tag{3}$$

$$\frac{\partial P(x,t)}{\partial t} = \gamma P(x,t) + \gamma x\frac{\partial P(x,t)}{\partial x} - \gamma x_0\frac{\partial P(x,t)}{\partial x} + D\frac{\partial^2 P(x,t)}{\partial x^2}$$

The Fourier transform is

$$\begin{aligned}
\frac{\partial \widetilde{P}(k,t)}{\partial t} &= \gamma\widetilde{P}(k,t) - \gamma\widetilde{P}(k,t) - \gamma k\frac{\partial\widetilde{P}(k,t)}{\partial k} - i\gamma x_0 k\widetilde{P}(k,t) - k^2 D\widetilde{P(k,t)} \\
\implies \frac{\partial\widetilde{P}(k,t)}{dt} &+ \gamma k\frac{\widetilde{P}(k,t)}{dk} + \gamma\big(k^2D + ikx_0\big)\widetilde{P}(k,t) = 0
\end{aligned}$$

We can use the method of characteristics to solve this PDE.

$$\frac{d\widetilde{P}}{dt} = \frac{\partial\widetilde{P}}{\partial t}\frac{ds}{dt} + \frac{\partial\widetilde{P}}{\partial k}\frac{dk}{ds}$$

Let

$$\begin{aligned}
\frac{ds}{dt} = 1; &\quad \frac{dk}{ds} = \gamma k \\
s = t; &\quad k = k_0 e^{\gamma t}
\end{aligned}$$

(Remaining proof under LaTeX construction)

$$\boxed{P(x,t) = \sqrt{\frac{1}{2\pi\big(\frac{D}{\gamma}(1-e^{-2\gamma t}) + w^2e^{-2\gamma t}\big)}}exp\left[-\frac{(x-x_0(1-e^{-\gamma t}))^2}{2\pi\big(\frac{D}{\gamma}(1-e^{-2\gamma t}) + w^2e^{-2\gamma t}\big)}\right]}$$

Since the answer is a Gaussian, one can read off the values of $\langle m \rangle$ and $\langle m^2 \rangle$

$$
\begin{aligned}
\langle m \rangle &= x_0(1 - e^{-\gamma t}) \\
\langle m^2 \rangle &= x_0^2(1 - e^{-\gamma t})^2 + \frac{D}{\gamma}(1 - e^{-2\gamma t}) + w^2 e^{-2\gamma t}
\end{aligned}
$$