Homework 7 - Due December 6, 2016

Given Eq.(i): $\frac{dz}{dt} + \Omega z = 0$, where $\Omega = \alpha + i\beta$ and $z = x + iy$, with $\alpha, \beta, x, y \in \mathbb{R}$ and $\Omega, z \in \mathbb{C}$, define a projection operator $\mathbb{P}$ such that $\mathbb{P}\Theta = \text{Re}(\Theta)$ and $(\mathbb{1} - \mathbb{P})\Theta = \text{iIm}(\Theta)$, where Θ is an arbitrary complex number. Find a closed equation for x(t).

Professer Kenkre's clarifications:

1. Defining appropriate projection operators and applying them a la Zwanzig to Eq.(i) derive an equation for the time evolution of x alone (real part of z) containing a proper term, a memory term and an initial term.
2. Assume an intial condition which will eliminate the initial term and call the resultant eq, closed in x(t), Eq.(ii).
3. Through EXACT manipulations of the Zwanzig kind simplify Eq.(ii) by eliminating the projection operators. Make no approximations!
4. Recognize the x evolution equation as something you recognize in physics and solve it explicitly. Comment.

Solution

Let $\mathbb{P}\Theta = \text{Re}(\Theta)$ and $(\mathbb{1} - \mathbb{P})\Theta = \text{iIm}(\Theta)$, where Θ is an arbitrary complex number. By applying the projection operators $\mathbb{P}$ and $(\mathbb{1} - \mathbb{P})$ to our original differential equation, we get:

$$\begin{aligned}
\mathbb{P}\left(\frac{dz}{dt} + \Omega z\right) &= \frac{dx}{dt} + \mathbb{P}(\alpha z - \beta y + i\alpha y + ix\beta) \\
&= \frac{dx}{dt} + \alpha y - \beta y = 0 \\
(\mathbb{1} - \mathbb{P})\left(\frac{dz}{dt} + \Omega z\right) &= i\frac{dy}{dt} + i(\alpha y + \beta x) = 0 \\
&\Rightarrow i\frac{dy}{dt} + i\alpha y = -i\beta x \\
&\Rightarrow y(t) = y(0)e^{-\alpha t} - \beta \int_0^t dt' e^{-\alpha(t-t')} x(t')
\end{aligned}$$

It follows that

$$\frac{dx}{dt} + \alpha x - \beta y(0)e^{-\alpha t} + \beta^2 \int_0^t dt' e^{-\alpha(t-t')} x(t') = 0$$

Assume y(0) = 0 so that

$$\boxed{\frac{dx}{dt} + \alpha x + \beta^2 \int_0^t dt' e^{-\alpha(t-t')} x(t') = 0}$$

If we mimic the Zwanzig approach:

$$\frac{d\mathbb{P}z}{dt} + \mathbb{P}\Omega\big[\mathbb{P} + (\mathbb{1} - \mathbb{P})\big]z = 0 \tag{1}$$

$$\frac{d(\mathbb{1} - \mathbb{P}))z}{dt} + (\mathbb{1} - \mathbb{P})\Omega\big[(\mathbb{1} - \mathbb{P}) + \mathbb{P}\big]z = 0 \tag{2}$$

$$(1) \Rightarrow \frac{d\mathbb{P}z}{dt} + \mathbb{P}\Omega\mathbb{P}z + \mathbb{P}\Omega(\mathbb{1} - \mathbb{P})z = 0$$

$$(2) \Rightarrow \frac{d(\mathbb{1} - \mathbb{P}))z}{dt} + (\mathbb{1} - \mathbb{P})\Omega(\mathbb{1} - \mathbb{P})z + (\mathbb{1} - \mathbb{P})\Omega\mathbb{P}z = 0$$

The solution to Eq. (2) is

$$(\mathbb{1} - \mathbb{P})z(t) = e^{-(\mathbb{1}-\mathbb{P})\Omega t}(\mathbb{1} - \mathbb{P})z(0) - \int_0^t dt' e^{-(\mathbb{1}-\mathbb{P})\Omega(t-t')}(\mathbb{1} - \mathbb{P})\Omega\mathbb{P}z(t')$$

Insert this into Eq. (1) to obtain

$$\Rightarrow \frac{d\mathbb{P}z(t)}{dt} + \mathbb{P}\Omega\mathbb{P}z + \mathbb{P}\Omega e^{-(\mathbb{1}-\mathbb{P})\Omega t}(\mathbb{1} - \mathbb{P})z(0) - \mathbb{P}\Omega\int_0^t dt' e^{-(\mathbb{1}-\mathbb{P})\Omega(t-t')}(\mathbb{1} - \mathbb{P})\Omega\mathbb{P}z(t') = 0$$

The first term:

$$\frac{d\mathbb{P}z}{dt} = \frac{dx}{dt}$$

The second term:

$$\mathbb{P}\Omega\mathbb{P}z = \alpha x$$

The third term:

$$\begin{aligned}
\mathbb{P}\Omega e^{-(\mathbb{1}-\mathbb{P})\Omega t}(\mathbb{1} - \mathbb{P})z(0) &= \mathbb{P}\Omega e^{-(\mathbb{1}-\mathbb{P})\Omega t} iy(0) \\
&= \mathbb{P}\Omega \sum_{n=0}^{\infty} \frac{(-1)^n t^n \big[(\mathbb{1} - \mathbb{P})\Omega\big]^n}{n!} iy(0)
\end{aligned}$$

Aside:

$$\begin{aligned}\big[(\mathbb{1}-\mathbb{P})\Omega\big]^n iy(0) &= \big[(\mathbb{1}-\mathbb{P})\Omega\big]^{n-1}(\mathbb{1}-\mathbb{P})\Omega iy(0)\\ &= \big[(\mathbb{1}-\mathbb{P})\Omega\big]^{n-1} i\alpha y(0)\\ &= \big[(\mathbb{1}-\mathbb{P})\Omega\big]^{n-2} i\alpha^2 y(0)\\ &\vdots\\ &= i\alpha^n y(0)\end{aligned}$$

That means that the third terms becomes

$$\mathbb{P}\Omega\sum_{n=0}^{\infty}\frac{(-1)^n t^n\alpha^n}{n!}iy(0) = -\beta y(0)e^{-\alpha t}$$

Assume $y(0) = 0$.

The forth term:

$$\begin{aligned}\mathbb{P}\Omega\int_0^t dt' e^{-(\mathbb{1}-\mathbb{P})\Omega(t-t')}(i\beta x(t')) &= \int_0^t dt'\sum_0^{\infty}\frac{(-1)^n(t-t')^n}{n!}\mathbb{P}\Omega i\alpha^n\beta x\\ &= -\beta^2\int_0^t dt' e^{-\alpha(t-t')}x(t')\end{aligned}$$

Putting them together:

$$\boxed{\frac{dx}{dt}+\alpha x+\beta^2\int_0^t dt' e^{-\alpha(t-t')}x(t') = 0}$$

To solve this equation, we can use Laplace Transforms:

$$\mathcal{L}\left[\frac{dx}{dt}\right] = -\alpha\mathcal{L}\big[x\big]-\beta^2\mathcal{L}\left[\int_0^t dt' e^{-\alpha(t-t')}x(t')\right] \tag{3}$$

Note, this is a convolution

$$\mathcal{L}\big[(f*g)(t)\big] = \mathcal{L}\left[\int_0^t dt' f(t-t')g(t')\right] = \tilde{f}(\epsilon)\tilde{g}(\epsilon)$$

Since

$$\mathcal{L}\big[e^{-\alpha t}\big] = \frac{1}{\epsilon+\alpha}$$

$$
\begin{aligned}
(3) &\Rightarrow \epsilon\tilde{x}(\epsilon) - x(0) = -\alpha\tilde{x}(\epsilon) - \beta^2 \frac{\tilde{x}(\epsilon)}{\epsilon+\alpha} \\
&\Rightarrow \tilde{x}(\epsilon)\left(\epsilon+\alpha+\frac{\beta^2}{\epsilon+\alpha}\right) = \tilde{x}(\epsilon)\left(\frac{(\epsilon+\alpha)^2+\beta^2}{\epsilon+\alpha}\right) = x(0) \\
&\Rightarrow \tilde{x}(\epsilon) = x(0)\left(\frac{\epsilon+\alpha}{(\epsilon+\alpha)^2+\beta^2}\right)
\end{aligned}
$$

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\mathcal{L}^{-1}\big[\tilde{x}(\epsilon)\big] = x(0)\mathcal{L}^{-1}\left[\frac{\epsilon+\alpha}{(\epsilon+\alpha)^2+\beta^2}\right]
$$

$$
\Rightarrow \boxed{x(t) = x(0)e^{-\alpha t}cos\beta t}
$$

This is a damped harmonic oscillator.