Some Basic Probability Concepts

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Contents

- Events and event operations
- Definition of probability
  - Classical probability (equally likely model)
  - Relative frequency probability
  - Probability properties
- Calculating probability of a event
  - Rule of complementation
  - Addition rule
  - Conditional probability
  - Multiplication rule
  - Independent event
  - Joint probability and marginal probability

Events

- Random experiment
  - An experiment (process or trial) with a random outcome
  - When being performed, it results in one of a well-defined collection (**sample space**) of possible chance outcomes (mutually exclusive, collectively exhaustive). Sub-collections of outcomes are called **events**
  - An event is said to occur if one of the outcomes contained in the event occurs
- Example 1
  - Tossing a simple six-sided die with faces marked with from one through six dots.
    - Sample space S = {1 dot, 2 dots, 3 dots, 4 dots, 5 dots, 6 dots}
    - A=“even number of dots” = {2 dots, 4 dots, 6 dots}
    - B=“number of dots >3” = {4 dots, 5 dots, 6 dots}

Favorable outcomes

Event – Example 2

- A group of students consists of six girls and three boys as follows:

ID	g1	g2	g3	g4	g5	g6	b1	b2	b3
Strong skill	dance	dance	singing	singing	piano	piano	dance	singing	piano
Wearing glasses	No	No	Yes	No	Yes	No	Yes	No	Yes

- Experiment: randomly select one as the representative of the group.
- If using ID of the selected student to denote the outcome, then the sample space can be written as

$$S = \{g1, g2, g3, g4, g5, g6, b1, b2, b3\}$$

- Event: the representative is
  - G = "a girl" = {g1, g2, g3, g4, g5, g6}
  - B = "a boy" = {b1, b2, b3}
  - P = "good at playing piano" = {g5, g6, b3}
  - W = "wearing glasses" = {g3, g5, b1, b3}

Favorable outcomes

Three Operations of Events: *not*, *or* and *and*

(Assuming S is sample space, A and B are events)

- Event (not *A*) = "*A* does not occur", is called the ***complement*** of *A*, consists of the outcomes in *S* that are not in *A*, often denoted by $\overline{A}$
  - In Example 1,

    A = "even number of dots" = { 2 dot, 4 dots, 6 dots }

    (not *A)* = "odd number of dots" = { 1 dot, 3 dots, 5 dots }
  - In Example 2,

    (not *G*) = *B* = {b1, b2, b3}

    (not *P*) = "good at dancing or singing"

    = {g1, g2, g3, g4, b1, b2}

ID	**g1**	**g2**	**g3**	**g4**	**g5**	**g6**	**b1**	**b2**	**b3**
Strong skill	dance	dance	singing	singing	piano	piano	dance	singing	piano
Wearing glasses	No	No	Yes	No	Yes	No	Yes	No	Yes

G = "a girl", *B* = "a boy", *P* = "good at playing piano", *W* = "wearing glasses"

Three Operations of Events: *not*, *or* and *and*

(Assuming S is sample space, A and B are events)

- Event (*A* or *B*) = "at least one of the events *A* and *B* occurs", is called the ***union*** of *A* and *B*, consist of all the outcomes that are in *A* or in *B* or in both, often denoted by *A* U *B*
  - In Example 1,

    A U *B* = "the number of dots is either even or is greater than 3"

    = {2 dots, 4 dots, 5, dots, 6 dots}

    (*A* = "even number of dots", *B* = "more than 3 dots")

  - In Example 2,

    W U *P* = "is either wearing glasses or good at playing piano"

    = {g3, g5, g6, b1, b3}

ID	g1	g2	g3	g4	g5	g6	b1	b2	b3
					P	P			P
Strong skill	dance	dance	singing	singing	piano	piano	dance	singing	piano
Wearing glasses	No	No	Yes	No	Yes	No	Yes	No	Yes

G = "a girl", *B* = "a boy", *P* = "good at playing piano", *W* = "wearing glasses"

Three Operations of Events: *not*, *or* and *and*

(Assuming S is sample space, A and B are events)

- Event (A and B) = "A and B occur simultaneously", is called the **intersection** of A and B, consists all the outcomes contains in both A and B, often denoted by $A \cap B$.
  - In Example 1,

    $A \cap B$ = "is an even number that is greater than 3"

    = {4 dots, 6 dots}
  - In Example 2,

    $W \cap P$ = "wearing glasses and good at playing piano as well"

    = {g5, b3}

ID	**g1**	**g2**	**g3**	**g4**	**g5**	**g6**	**b1**	**b2**	**b3**
Strong skill	dance	dance	singing	singing	piano	piano	dance	singing	piano
Wearing glasses	No	No	Yes	No	Yes	No	Yes	No	Yes

G = "a girl", B = "a boy", P = "good at playing piano", W = "wearing glasses"

Two Special Events

(Assuming S is sample space, A and B are events)

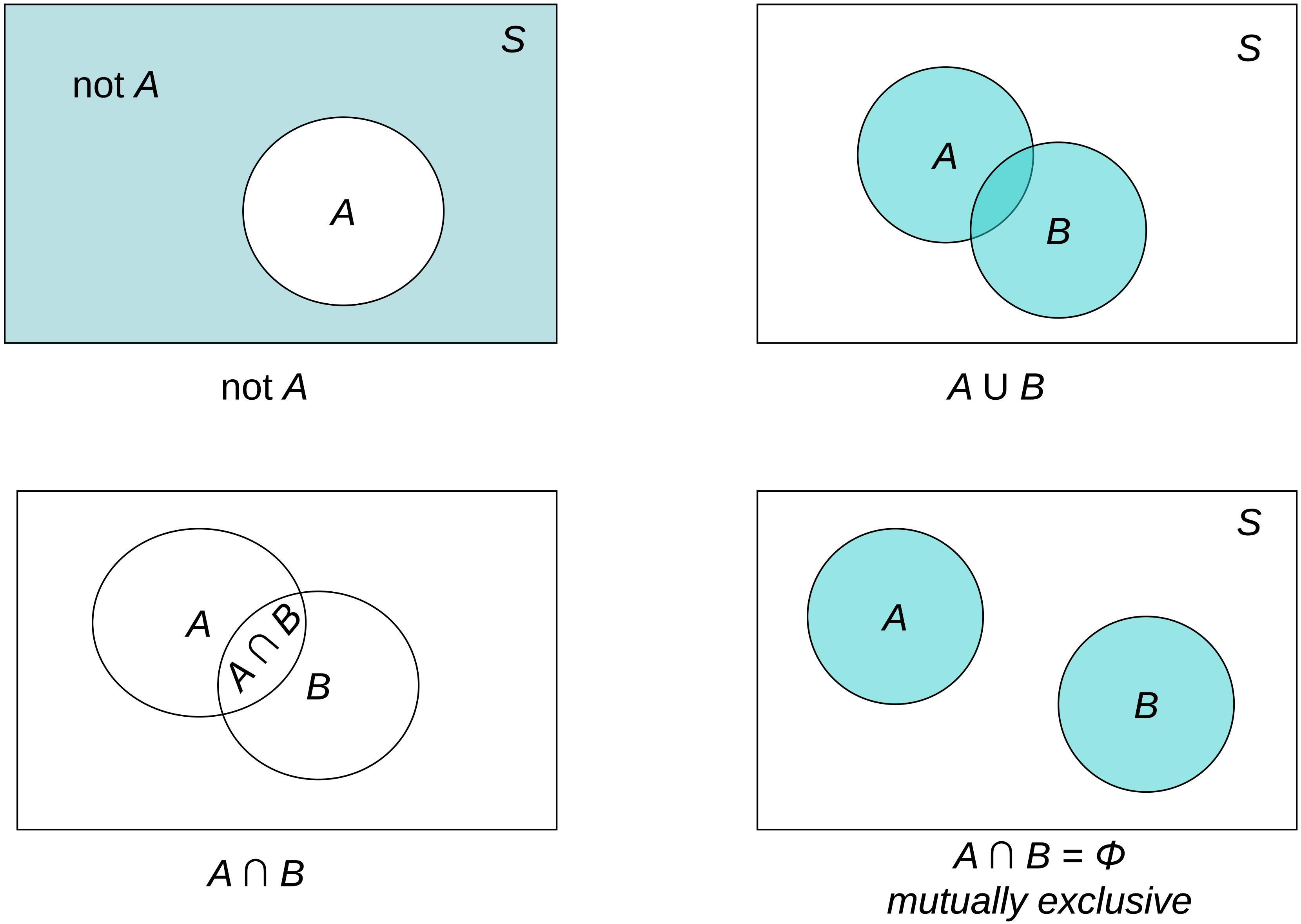
- S is called *sure* event. In Example 2, $G \cup B = S$
- Φ = the empty set, is called *null* event, in Example 2, $G \cap B = \Phi$

- Events G and B are mutually exclusive ($G \cap B = \Phi$) and collectively exhaustive ($G \cup B = S$).

ID	**g1**	**g2**	**g3**	**g4**	**g5**	**g6**	**b1**	**b2**	**b3**
Strong skill	dance	dance	singing	singing	piano	piano	dance	singing	piano
Wearing glasses	No	No	Yes	No	Yes	No	Yes	No	Yes

G = "a girl", B = "a boy", P = "good at playing piano", W = "wearing glasses"

Venn diagram showing the operations of events



Probability

- Classical probability (Equally likely model)
  - A random experiment has finite number (n) of (mutually exclusive and collectively exhaustive) outcomes. All the (n) outcomes are equally likely to occur. If event A consists of m of such outcomes. Then

$$P(A) = m / n$$

  - Example 1 $P(A) = 3/6 = 0.5$, where A = "even number of dots" = { 2 dot, 4 dots, 6 dots }

  - Example 2 $P(G) = 6/9 = 2/3$

$$P(P) = 3/9 = 1/3$$

ID	**g1**	**g2**	**g3**	**g4**	**g5**	**g6**	**b1**	**b2**	**b3**
Strong skill	dance	dance	singing	singing	piano	piano	dance	singing	piano
Wearing glasses	No	No	Yes	No	Yes	No	Yes	No	Yes

G = "a girl", B = "a boy", P = "good at playing piano", W = "wearing glasses"

Probability

- Relative Frequency Probability
  - If a random experiment is repeated a large number of times, n, and if a resulting event E occurs m times, then the relative frequency of occurrence of E, m / n, will be approximately equal to the probability of E, i.e.

$$P(E) \approx m / n$$

Elementary properties of probability

- $P(E) \geq 0$, $P(S) = 1$, $P(\Phi) = 0$.
- Suppose $E_1, E_2, \ldots, E_m$ are mutually exclusive and collectively exhaustive events. Then

  $$P(E_1) + P(E_2) + \ldots + P(E_\mathrm{m}) = 1$$

- For any mutually exclusive events E_i, E_j

  $$P(E_\mathrm{i} \cup E_\mathrm{j}) = P(E_\mathrm{i}) + P(E_\mathrm{j})$$

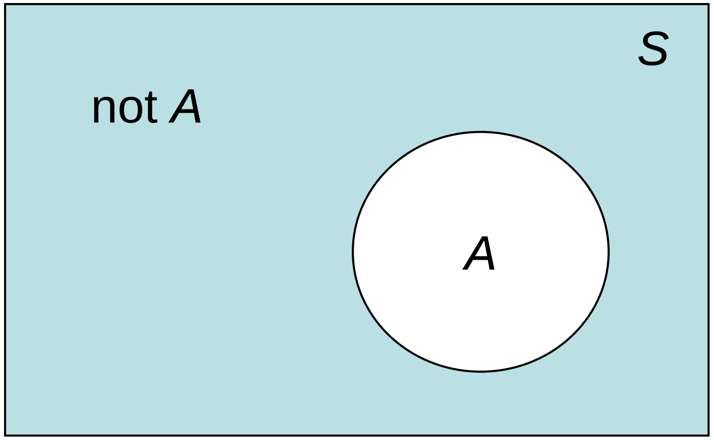
  - Note. “U” can be replaced by “+” for mutually exclusive events.

New understanding of probability: (1) measure the chance of event, (2) satisfies the three properties.

Calculating the probability of an event – Rule of Complementation

- $P(\text{not } A) = 1 - P(A)$
- Example 1
  - $P(\text{not } A) = 1 - 1/2 = 1/2$,
- Example 2
  - $P(B) = P(\textit{not } G) = 1 - P(\text{G}) = 1 - 2/3 = 1/3$
  - $P(\text{wearing glasses}) = P(W) = 4/9$
  - $P(\text{not wearing glasses}) = P(\text{not } W) = 1 - P(W)$

    $= 1 - 4/9 = 5/9$

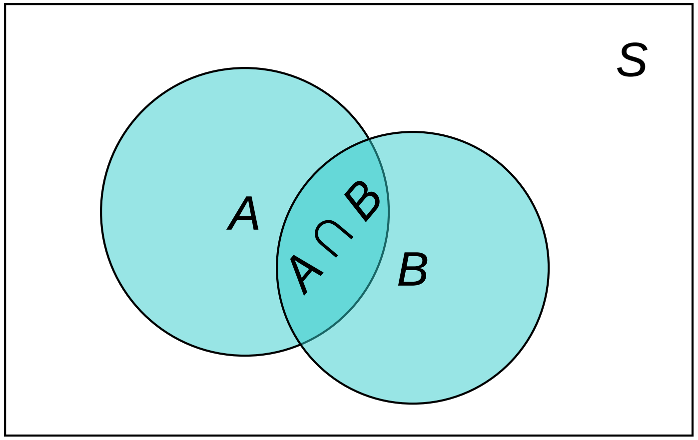


ID	**g1**	**g2**	**g3**	**g4**	**g5**	**g6**	**b1**	**b2**	**b3**
Strong skill	dance	dance	singing	singing	piano	piano	dance	singing	piano
Wearing glasses	No	No	Yes	No	Yes	No	Yes	No	Yes

G = "a girl", *B* = "a boy", *P* = "good at playing piano", *W* = "wearing glasses"

Calculating the probability of an event – Addition Rule

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



- Example 1
  - $P(A) = 1/2$, $P(B) = 1/2$, $P(A \cap B) = 2/6 = 1/3$,
  - $P(A \cup B) = 1/2 + 1/2 - 1/3 = 2/3$
    - A=“even number of dots” = {2 dots, 4 dots, 6 dots}
    - B=“number of dots >3” = {4 dots, 5 dots, 6 dots}

- Example 2
  - $P(G) = 6/9 = 2/3$, $P(P) = 3/9 = 1/3$, $P(G \cap P) = 2/9$,
  - $P(G \cup P) = 2/3 + 1/3 - 2/9 = 1 - 2/9 = 7/9$

ID	g1	g2	g3	g4	g5	g6	b1	b2	b3
Strong skill	dance	dance	singing	singing	piano	piano	dance	singing	piano
Wearing glasses	No	No	Yes	No	Yes	No	Yes	No	Yes

G = “a girl”, B = “a boy”, P = “good at playing piano”, W = “wearing glasses”

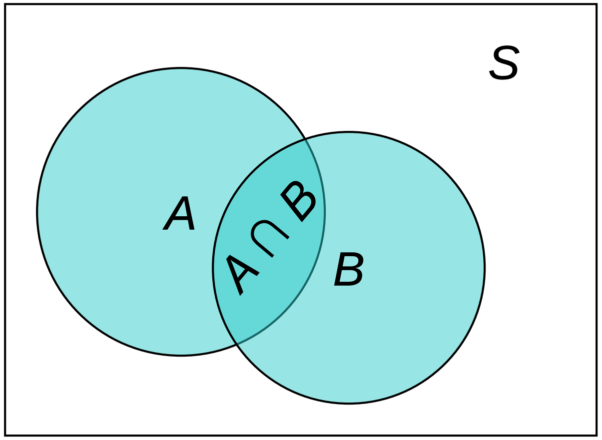
Calculating the probability of an event – Conditional Probability

- Example 2, Suppose we pick one at random from the 9 students and find that the student is a girl (*G*). What is the probability that this student is wearing glasses (*W*)?
- This is called the conditional probability of W given that *G* is known to have occurred and is denoted by $P(W|G)$.
- Question: $P(W|G) \overset{?}{=} P(W)$ (= 4/9)

$$P(W|G) = \frac{2}{6} = \frac{2/9}{6/9} = \frac{P(W \cap G)}{P(G)}$$

- If $P(A) \neq 0$, then $P(B|A)=P(B \cap A)/P(A)$

  (Conditional probability of *B* given *A*)



ID	g1	g2	g3	g4	g5	g6	b1	b2	b3
Strong skill	dance	dance	singing	singing	piano	piano	dance	singing	piano
Wearing glasses	No	No	Yes	No	Yes	No	Yes	No	Yes

G = "a girl", *B* = "a boy", *P* = "good at playing piano", *W* = "wearing glasses"

Calculating the probability of an event – Multiplication Rule

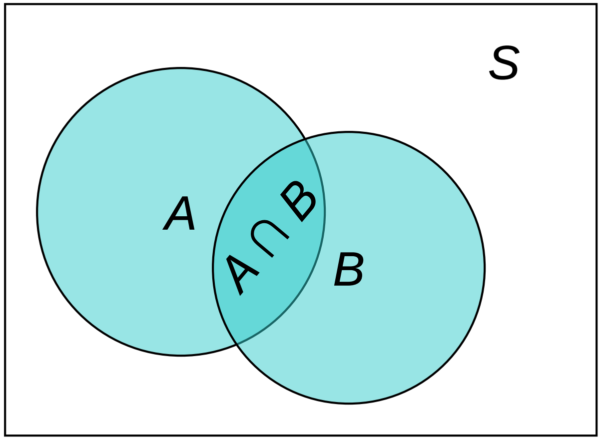
- Conditional Probability
  - If $P(A) \neq 0$, then $P(B|A)=P(B \cap A)/P(A)$

↓

Multiply both sides by $P(A)$

↓

$$P(A) \cdot P(B|A) = P(B \cap A)$$



- Conditional Probability
  - If $P(B) \neq 0$, then $P(A|B)=P(A \cap B)/P(B)$

↓

$$P(A) \cdot P(B|A) = P(B \cap A)$$

Multiplication Rule

$$P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$$

Calculating the probability of an event – Multiplication Rule

- $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$
- Example 3. An epidemiological study indicates that the prevalence of disease *A* is 0.001 and one out of four disease *A* patients also suffer disease *B*. What is the probability that one suffers both diseases *A* and *B* simultaneously?
  - $P(A) = 0.001$, $P(B|A) = 1/4$,
  - $P(A \cap B) = P(B|A)\, P(A) = 1/4 \times 0.001 = 0.00025$
- Now suppose the prevalence of disease *B* is 0.002. What is the chance that a disease *B* patient also suffers disease *A*?
  - $P(B) = 0.002$,
  - $P(A|B) = P(A \cap B)/P(B) = 0.00025/0.002 = 0.125$

Calculating the probability of an event – Independent Event

- We say that A and B are independent events if $P(A)=P(A|B)$.
- If A and B are independent, then

$$P(A \cap B) = P(A)P(B);\ P(A) \neq 0,\ P(B) \neq 0.$$

- The following statements are equivalent
  - $P(A \mid B) = P(A)$
  - $P(B \mid A) = P(B)$
  - $P(A \cap B) = P(A)P(B)$
- In Example 2, Consider event G and P
  - $P(G) = 2/3$, $P(P) = 3/9 = 1/3$, $P(P|G) = 2/6 = 1/3 = P(P)$

    Hence event P is independent of G
  - $P(G|P) = 2/3 = P(G)$, $P(P \cap \text{G}) = 2/9$, $P(P)P(G) = (2/3) \times (3/9) = 2/9$
  - In fact, event P is also independent of B = (not G), Why?
  - W and P ? $P(P)=3/9=1/3$, $P(P|W)=2/4=1/2 \neq P(P)$. W and P are not independent.

ID	**g1**	**g2**	**g3**	**g4**	**g5**	**g6**	**b1**	**b2**	**b3**
Strong skill	dance	dance	singing	singing	piano	piano	dance	singing	piano
Wearing glasses	No	No	Yes	No	Yes	No	Yes	No	Yes

G = "a girl", B = "a boy", P = "good at playing piano", W = "wearing glasses"

Calculating the probability of an event – Joint and Marginal Probabilities

Frequency of Gender by whether wearing glasses

Gender	Wearing Glasses: Yes (W)	Wearing Glasses: No (not W)	Total
Girl (G)	2	4	6
Boy (B)	2	1	3
Total	4	5	9

Probabilities of possible outcomes

Gender	Wearing Glasses: Yes (W)	Wearing Glasses: No (not W)	Total
Girl (G)	$P(G \cap W)$	$P(G \cap (\text{not } W))$	$P(G)$
Boy (B)	$P(B \cap W)$	$P(B \cap (\text{not } W))$	$P(B)$
Total	$P(W)$	$P(\text{not } W)$	1

Joint probabilities

Marginal probabilities

Calculating the probability of an event – Joint and Marginal Probabilities

Frequency of Gender by Student's Strong Skill

Gender	Strong Skill			Total
	Dancing (D)	Singing (C)	Piano (P)	
Girl (G)	2	2	2	6
Boy (B)	1	1	1	3
Total	3	3	3	9

Joint probabilities and marginal probabilities

Gender	Strong Skill			Total
	Dancing (D)	Singing (C)	Piano (P)	
Girl (G)	$P(G \cap D) = 2/9$	$P(G \cap C) = 2/9$	$P(G \cap P) = 2/9$	$P(G) = 6/9$
Boy (B)	$P(B \cap D) = 1/9$	$P(B \cap C) = 1/9$	$P(B \cap P) = 1/9$	$P(B) = 3/9$
Total	$P(D) = 1/3$	$P(C) = 1/3$	$P(P) = 1/3$	1

Partition of sample space $S = G + B$, also $S = D + C + P$

$P(G) = P(G \cap D) + P(G \cap C) + P(G \cap P)$

$P(D) = P(G \cap D) + P(B \cap D), \ldots$ – Law of total Probability

Calculating the probability of an event – Joint and Marginal Probabilities

(Example 3.4.1 on Page 69, 10^{th} Ed.)

Family history	Early ≤ 18 (E)	Later > 18 (L)	Total
Negative (A)	28	35	63
Bipolar Disorder (B)	19	38	57
Unipolar (C)	41	44	85
Unipolar & Bipolar (D)	53	60	113
Total	141	177	318

- Study to investigate the effect of the age at onset of bipolar disorder on the course of the illness.
- Variables of interest
  - Family history of mood disorders: Negative, Bipolar Disorder, Unipolar, Unipolar & Bipolar.
  - Groups related to age at onset of disorder: Early ≤18, Later > 18
- Experiment
  - Pick one person at random from the sample and look at the possible events (outcome) of interest and their probabilities

Calculating the probability of an event – Joint and Marginal Probabilities

Joint probabilities and marginal probabilities

(Example 3.4.1 on Page 69)

Family history	Early ≤ 18 (E)	Later > 18 (L)	Total
Negative (A)	$P(A \cap E)$	$P(A \cap L)$	$P(A)$
Bipolar Disorder (B)	$P(B \cap E)$	$P(B \cap L)$	$P(B)$
Unipolar (C)	$P(C \cap E)$	$P(C \cap L)$	$P(C)$
Unipolar & Bipolar (D)	$P(D \cap E)$	$P(D \cap L)$	$P(D)$
Total	$P(E)$	$P(L)$	1.00

Joint probabilities

Marginal probabilities

Partition of sample space $S = E + L$, also $S = A + B + C + D$

$P(E) = P(A \cap E) + P(B \cap E) + P(C \cap E) + P(D \cap E)$,

$P(A) = P(A \cap E) + P(A \cap L), \ldots$

– Law of total Probability

Summary

- What we learned
  - Events and event operations
  - Definition of probability
    - Classical probability (equally likely model)
    - Relative frequency probability
    - <u>Probability properties</u>
  - Calculating probability of a event
    - Rule of complementation
    - Addition rule
    - <u>Conditional probability</u>
    - Multiplication rule
    - <u>Independent event</u>
    - Joint probability and marginal probability
- Focus on understanding of the concepts, especially, those with the underscores.

Home work

(Due Friday, July 24)

- Chapter 3 Exercises 3.4.2, 3.4.3 on page 77 (10th Ed.) (Choose one subject from the study sample and look at the possible outcomes and their probabilities)
- Chapter 3 Exercises 3.4.7 on page 78.
- Chapter 3 Review questions and exercises 5 on page 87, and18 – 20 on page 89