Pooled Standard Error versus the Satterthwaite Approximation

In this segment we'll clarify the differences between the two methods of accounting for two sample variances by running through the equations in a little more detail. The Pooled Standard error is used when the variances between the two samples are equivalent:

$$se_P = s_P\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} \qquad (1)$$

where, if the sample sizes are the same $s_P^2 = \frac{s_1^2 + s_2^2}{2}$, (2)

which is the same as writing $s_P = \frac{s_1 + s_2}{2}$ as $s_P = \sqrt{s_P^2}$

or equivalently, if the sample sizes are different, $s_P^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$, (3)

so $s_P^2 = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$.

Let's say we have two samples:

Sample 1 = $(\overline{Y}_1 = 15, s_1 = 5, n_1 = 25)$
Sample 2 = $(\overline{Y}_2 = 15, s_2 = 5, n_2 = 25)$

So, our $s_P = 5$ from equation 2, and plugging this into equation 1 we find:

$$se_P = 5\sqrt{\frac{1}{25}+\frac{1}{25}} = 5\sqrt{0.08} = 1.414214$$

The Satterthwaite approximation of the standard errors differs from the Pooled method in that it does not *assume* that the variances of the two samples are equal. This means that if the variances are equal, the Satterthwaite approximation should give us exactly the same answer as the Pooled method. To demonstrate this, let's run the above sample statistics through the Satterthwaite approximation.

$$se_S = \sqrt{\frac{s_1^2}{n_2}+\frac{s_2^2}{n_2}} = \sqrt{\frac{25}{25}+\frac{25}{25}} = \sqrt{2} = 1.414214$$

Now, let's change the sample statistics a slight bit to demonstrate the different behaviors of the two equations:

Sample 1 = $(\overline{Y}_1 = 15, s_1 = 5, n_1 = 30)$

Sample 2 = $\left(\overline{Y}_2 = 15, s_2 = 5, n_2 = 25\right)$

So, now the sample sizes are different but the variances are still equal. Putting these statistics through the two equations:

$$se_P = 5\sqrt{\frac{1}{30}+\frac{1}{25}} = 5\sqrt{0.073\overline{3}} = 1.354006\text{, and}$$
$$se_S = \sqrt{\frac{25}{30}+\frac{25}{25}} = \sqrt{0.83\overline{3}+1} = 1.354006$$

We find that, again, both methods are equivalent, and this is because both methods account for the different sample sizes in exactly the same way, the denominator of the squared terms. Now let's vary the variances:

Sample 1 = $\left(\overline{Y}_1 = 15, s_1 = 8, n_1 = 25\right)$
Sample 2 = $\left(\overline{Y}_2 = 15, s_2 = 5, n_2 = 25\right)$

For clarity's sake we have made the sample sizes equivalent again. Let's start with the Satterthwaite approximation this time:

$$se_S = \sqrt{\frac{64}{25}+\frac{25}{25}} = \sqrt{2.56+1} = 1.886796\text{, and as in the Pooled method}$$

$$s_P = \frac{s_1+s_2}{2} = \frac{8+5}{2} = 6.5\,,$$

$$se_P = 6.5\sqrt{\frac{1}{25}+\frac{1}{25}} = 6.5\sqrt{0.08} = 1.838478\,.$$

So, in this case we see that the Pooled method under-estimates the real standard error given by the Satterthwaite approximation when the variances are not equal. This is because the first term of the Pooled method takes the arithmetic mean of the standard deviations (or variances), whereas, what we really need is a of *weighted* average.

In summary, when we talk about accounting for both variances, the difference between the two methods is really about how we treat the standard deviations: in the Pooled method, we are taking the arithmetic average of the standard deviations and converting this value into a standard error, whereas in the Satterthwaite approximation we are calculating the standard error from the weighted average of the two variances, a subtle, but important difference. Remember, the main difference is that the Satterthwaite approximation does not assume equal variances, whereas the Pooled method does. In other words, you can *always* use the Satterthwaite method and be correct, but you can only use the Pooled method in very specific (and rare) circumstances.