

Poisson Statistics v2.1

Background. The Poisson distribution can characterize random events that occur at a well-defined average rate. It is widely used in atomic and sub-atomic physics.

The Poisson distribution is effective in a variety of statistical applications. The most common involve event probabilities, but several assumptions must hold true: i) the rate at which random events occur does not change for the duration of the measurement; ii) the occurrence of one event does not change the likelihood of another event; iii) events occur at a slow enough rate that they can be individually distinguished.

In this experiment, Poisson statistics will be used to analyze random radioactive decay events that occur in a defined time interval. A decay occurs an integer k number of times in the interval, including possibly not at all ($k = 0$). The average number of events expected in a defined time interval is λ , known as the event rate. Given λ , the probability of observing k events in the time interval is:



Siméon Denis Poisson
1781–1840

$$P_k = e^{-\lambda} \frac{\lambda^k}{k!} \quad (1)$$

If the interval gets longer, λ increases commensurately and P_k changes.

Experiment. The radioactive source is Cs-137, with half-life of 30.2 years. The probability that a single atom will decay in a 100 ms interval, for example, is 6.87×10^{-11} , i.e. an extremely unlikely occurrence. If the source has enough mass, however, a sufficient number of decays will happen in a reasonable measurement time.

Because their energy is so high, it is difficult to detect gamma-rays directly. Indirect detection is used here through a process called *scintillation*. A scintillator is a special material – in this case a sodium iodide (NaI) crystal – that converts high-energy photons such as gamma-rays into visible photons that can be detected. The NaI scintillation crystal is directly attached to a photo-multiplier tube (PMT) that is sensitive enough to resolve very few visible photons. Detected photons appear as voltage pulses at the output of the PMT. The height of the voltage pulse is directly proportional to the energy of the gamma-ray that created it, which is 662 keV for Cs-137.

A multi-channel analyzer (MCA) can measure the energy of the gamma-rays and also the rate at which they strike the scintillator. The rate of spontaneous gamma-ray emission is of interest in this experiment. The MCA is configured to count gamma-rays in a defined time interval. It does this counting repetitively, building up a statistical ensemble of N intervals each holding an event count. When viewed as a histogram, a distinct distribution becomes evident.

The key to observing a Poisson distribution is keeping the event rate λ sufficiently low. The distribution plot will appear asymmetric. As λ increases, the distribution transitions to a symmetric Gaussian.

Equipment and Setup. The PMT/scintillator and radioactive source must be shielded inside a lead brick enclosure. Use caution when moving the bricks as they are very heavy. Radioactive sources are stored in a locked box. Ask the instructor for access.

The MCA is a model UCS30 with accompanying GUI software and user manual. Referring to the user manual will make setup and data taking easier. The most critical aspect of setup is the PMT high-voltage setting. The polarity and maximum voltage will be marked on the PMT (typically +1200V). Please have your setup confirmed by an instructor before enabling high-voltage.

Initial data is acquired *without* a radioactive source. Place the PMT/scintillator inside the lead brick enclosure and proceed to the MCA setup.

The MCA communicates with a host PC via USB. It biases the PMT using a special BNC cable with high voltage connectors. Never force a BNC cable onto a non-mating connector. A second BNC cable feeds the output of the PMT to the MCA, where pulses are appropriately amplified, conditioned, and counted.

Quick-start for the UCS-30 is as follows. From the menu bar, select Spectrum: Connect to Device, then Mode: Pulse Height Analysis (Preamp-in). In the main menu, set the High Voltage to 1000V and click the OFF button to turn the HV ON. Under Settings select Amp/HV/ADC. Set Conversion Gain at 256; Coarse Gain: 1; Fine Gain: 1. Make sure the discriminator range is at maximum: 3–256.

Switch to Mode: MCS Internal. The duration of the acquisition interval is set with Settings: MCS Settings: Dwell Time. The combination of the adjustable PMT voltage (i.e. gain) and dwell time will determine the event rate. The MCA acquires counts in a succession of $N = 256$ measurement intervals.

Press the run button. Counts will be displayed for each interval. This should be a relatively flat line, but with statistical fluctuations. You will need to make 3 data runs to attain 3 different average event rates in each interval: i) 1–2 counts, ii) 5 counts, and iii) 10 counts. At the lowest event rate, there will be many empty intervals. At the highest, there will likely be none. To get the needed count rates, perform trial-and-error experimentation with the PMT voltage and dwell time. When an acceptable data run has been completed, save it to disk as a tab or comma delineated file. In your writeup, discuss why counts would be present in the absence of a radioactive source.

Place a 1 μCi Cs-137 source next to the scintillator. Set Mode: Pulse Height Analysis (Preamp-in). In the main menu, set the High Voltage. This is not critical; 1000V is a good initial setting. Under Settings select Amp/HV/ADC. Set Conversion Gain at 256; Coarse Gain: 4; Fine Gain: 1. Click the OFF button to turn the HV ON. Press the run button, let counts accumulate, and identify the prominent photo-peak. This peak corresponds to the 662 keV gamma-ray of Cs-137. The combination of high-voltage and gain determines where

the peak appears in the MCA display. If these values are too high, the rate of gamma-ray detection will be too rapid to observe Poisson statistics. Adjust the discriminator high and low channel settings to bracket this peak. This can be done from the menu or by sliding the two triangles below the x-axis. Note: This is not the same as setting the ROI.

Switch to Mode: MCS Internal. The voltage and gain should no longer be adjusted to change the event rate since the discriminator levels will no longer be correct. The rate can be modified with the dwell time as before and also by changing the distance between the Cs-137 source and scintillator. If the voltage/gain must be changed to get an acceptable event rate, the discriminator will also have to be reset in the PHA mode.

Make 3 data runs as before to attain 3 different average event rates in each interval: i) 1–2 counts, ii) 5 counts, and iii) 10 counts. Save data to disk as a tab or comma delineated file.

Analysis. Using software of your choice, plot a histogram for each of your 6 data sets. Make sure the x-axis bins are configured for integer counts. For sufficiently low event rates, the plotted distribution should be asymmetric and skewed towards the origin, consistent with Poisson statistics.

There are statistical fluctuations associated with this measurement. These are distinguished from fluctuations due to *instrumental* uncertainty, for example, variation of the MCA measurement intervals. The statistical error can be obtained by doing this measurement repeatedly, but it is much easier to realize that the count in each histogram bin originates from random events. If C_k is the count in the k -th bin, the uncertainty can be represented by the standard deviation σ_k associated with Poisson statistics: $\sigma_k = \sqrt{C_k}$. This is an important feature of Poisson statistics compared to a Gaussian distribution. In the latter, the standard deviation must be determined independently. Add statistical error bars σ_k to each point of your histogram.

To generate a theoretical curve, the average event rate λ is required. This should be calculated for each data set. This is not simply using the approximate target rates of 1–2, 5, 10 counts/interval, although the target rate may be similar to λ . A precise number for λ must be obtained from the data. With Equation (1), the total number of event intervals N , and the extracted value of λ , a Poisson curve can be generated and compared to the experimental result. The standard deviation of the distribution is $\sigma = \sqrt{\lambda}$.

How does the average event rate $\lambda \pm \sigma$ compare to the counts C_k in the N individual intervals? Can you quantify how well this rate describes what was measured in the ensemble of intervals?

The quality or “goodness” of the Poisson curve in describing the distribution is quantified with the parameter χ^2 . If there are n bins in the histogram, then

$$\chi^2 = \sum_k^n \frac{[C_k - P_k N]^2}{\sigma_k^2} = \sum_k^n \frac{[C_k - P_k N]^2}{C_k} \quad (2)$$

This characterizes the difference between experiment (C_k) and theory ($P_k N$) at each point

on the histogram (numerator) and the expected spread (denominator). $\chi^2 = 0$ represents perfect agreement, but this should not happen with a statistical ensemble. Intuitively, we expect that experimental and theoretical deviations to be about the same, i.e. $\chi^2 \approx n$. The expectation value is reduced by the number of degrees of freedom, which is one in this case, leading to: $\langle \chi^2 \rangle = n - 1$. Perform this analysis on your data and compare to the expected result. Further discussion can be found in the textbook by Bevington and Robinson on data reduction and error analysis.

At higher event rates, the histograms become more symmetric and are better described by a Gaussian distribution:

$$G_k = \frac{1}{\sqrt{2\pi\lambda}} \exp \left[\frac{-(k - \lambda)^2}{2\lambda} \right] \quad (3)$$

Evaluate Eq. (3) for the three data sets without Cs-137. The Gaussian curves G can be compared to the Poisson curves P that were obtained from Eq. (1). To do this, normalize both curves, then calculate and plot the difference curve: $(G - P)/P$. Compare by plotting this curve together with the predicted result T_k :

$$T_k = \frac{\delta - \delta^3/3\lambda}{2\lambda} \quad (4)$$

where $\delta = k - \lambda$. This provides a clear visualization of the asymmetry. More information can be found in the 1975 paper by L.J. Curtis on the class website.