Summer study course on many-body quantum chaos

Pablo Poggi

Center for Quantum Information and Control (CQuIC)

University of New Mexico

CQuIC
Center for Quantum Information and Control

QUANTUM SYSTEMS ACCELERATOR
Catalyzing the Quantum Ecosystem

Q-SEnSE
An NSF Quantum Leap Challenge Institute

Session 1: Overview of classical mechanics and Hamiltonian chaos

Wednesday June 2nd 2021

Background picture: Tent Rocks, New Mexico (USA)

This course

- **Format**: self-study. Not a class. Not a seminar.
- Participants read materials about the topics they have chosen, and then use what they learned to introduce the topic and lead a discussion about it.
- **Goals**: learn about the fundamentals and get a taste of new developments in a fast-moving field. Accumulate material for future reference.
- Start at ~ 4 pm MDT – we go until 5.15 pm MDT including discussion
- Schedule and materials available online at http://www.unm.edu/~ppoggi/
- Presentations will be recorded each week
  - UNM people can access recordings via **Stream** using their NetID
  - People outside of UNM can access a shared folder with the recordings via this **link** (password required)

Session	Date	Topic	Presenter
1	June 2nd	**Integrability and chaos in classical systems**	Pablo Poggi
2	June 9th	**Introduction to quantum chaos**	Pablo Poggi
3	June 16th	**Random Matrix Theory**	Changhao Yi
4	June 23rd	**Quantum integrability**	Manuel Muñoz
5	June 30th	**Thermalization in closed quantum systems**	Sam Slezak / Mason Rhodes
6	July 7th	**Scrambling and OTOCs I**	Sivaprasad Omanakuttan
7	July 14th	**Scrambling and OTOCs II**	Tyler Thurtell / Conor Smith
8	July 21st	**Random unitary evolution**	Andrew Zhao
9	July 28th TBD	**Many body localization and quantum scars**	Anupam Mitra / Karthik Chinni
10	July 28th TBD	**Dual unitary circuits**	Jun Takahashi + others

Quantum chaos: why?

Chaos in classical dynamical systems - deterministic randomness and *butterfly effect*

Extreme sensitivity of trajectories to changes in initial conditions

1

Quantum systems

→ No trajectories, observables evolve quasi-periodically in time. No obvious characterization of chaos

→ Correspondence: generic properties of quantum systems whose classical counterpart is chaotic (level statistics, eigenstate properties, semiclassical analysis, connection to random matrix theory)

2,3

1980's: G. Casati, M. Berry, O. Bohigas, M. Giannoni, C. Schmit 'Quantum Chaology'

Complete characterization of quantum chaos?

- Dynamics: connection to ergodicity and thermalization in statistical mechanics **5**
- Include systems without a well-defined classical limit. Quantum integrability. **4, 9, 10**
- Properties of quantum features (i.e. entanglement) in generic (nonintegrable) quantum many body systems

Quantum information

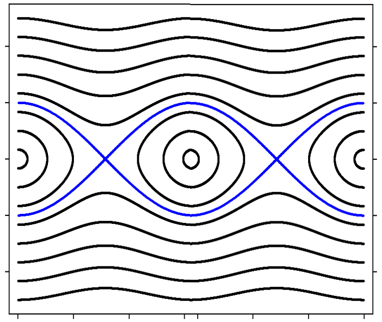
Dynamics of interacting quantum systems out of equilibrium → Hard to simulate classically (in general); Quantum computers to study quantum chaos!

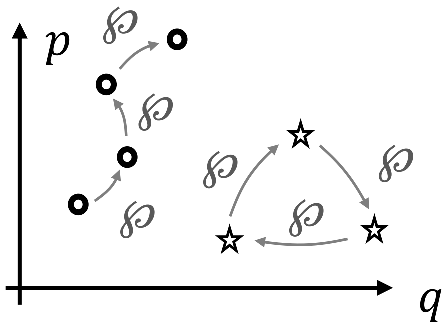
Sensitivity of quantum states to external perturbations → Reliable quantum information processing; Quantum metrology **6,7**

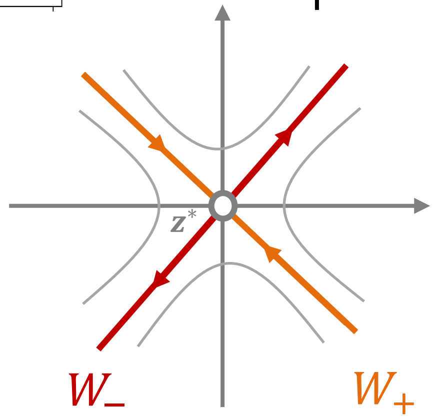
Dynamics of quantum information and generation of randomness → Scrambling and random circuits **8**

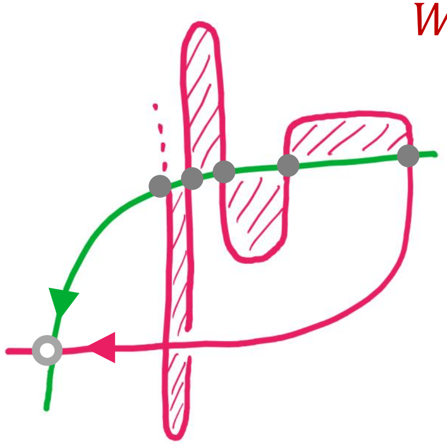
Today’s menu

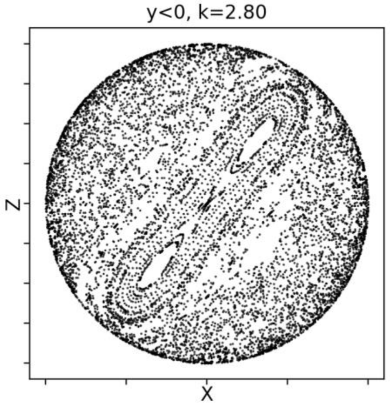
1. **Classical mechanics and Hamiltonian formalism**
2. **Integrability in classical systems and KAM theorem**
3. **Area-preserving maps and transition to chaos**
4. **Lyapunov exponents**
5. **Ergodicity and mixing**











Today’s presentation is mostly based on **“Nonlinear dynamics and quantum chaos: an Introduction” – by S. Wimberger**

Hamiltonian systems

- Lagrangian formalism $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \longrightarrow \mathbf{q} \in$ "configuration space" $dim(\mathbf{q}) = n$
- Hamiltonian formalism $H(\mathbf{q}, \mathbf{p}, t) \longrightarrow \mathbf{z} = (\mathbf{q}, \mathbf{p}) \in$ **"phase space"** $dim(\mathbf{z}) = 2n$

$n = \#\ \mathrm{d.o.f.}$ (degrees of freedom)

Equations of motion
$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases}$$

Or, in fancy form: $\dot{\mathbf{z}} = J \nabla H$

Symplectic matrix $J = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$

Autonomous and non autonomous systems

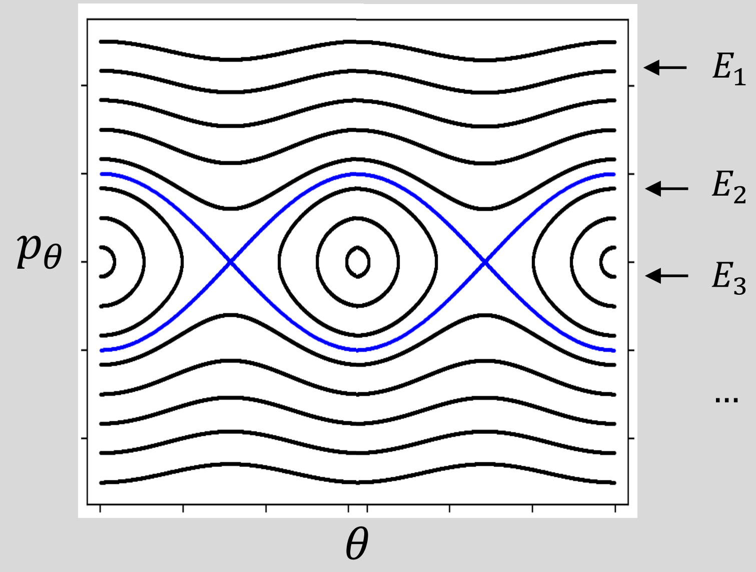
- $H(q,p)$ time-independent $\Rightarrow H\big(q(t), p(t)\big) = H\big(q(0), p(0)\big) = E$
- $H(q,p,t) = E(t)$ explicitly time-dependent
  - Mappable to an **autonomous** system in an extended phase space
  - Can be visualized using **surfaces of section** (coming in a few slides)

Liouville's theorem: time-evolution of a Hamiltonian system preserves phase space volume

e.g. Particle in 1D $H = \frac{p^2}{2m} + V(q)$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial V}{\partial q}$$

Pendulum $H(\theta, p_\theta) = \frac{p_\theta^2}{2m} - mgL\ cos\ \theta$



Canonical transformations

- I can take my original Hamiltonian and **transform coordinates** $(\mathbf{q},\mathbf{p}) \xrightarrow{\mathcal{M}} (\mathbf{Q}(\mathbf{q},\mathbf{p},\mathbf{t}),\mathbf{P}(\mathbf{q},\mathbf{p},\mathbf{t}))$

$$(\mathbf{q},\mathbf{p}) \downarrow H(\mathbf{q},\mathbf{p}) \qquad (\mathbf{Q},\mathbf{P}) \downarrow H'(\mathbf{Q},\mathbf{P})$$

- Relevant transformations are those which **preserve** the form of Hamilton's EOMs: $\dot{Q}_k = \frac{\partial H'}{\partial P_k} \qquad \dot{P}_k = -\frac{\partial H'}{\partial Q_k}$

↓

Canonical transformations

- Transformation $\mathcal{M}$ is **canonical** if and only if $\{Q_i,Q_j\} = \{P_i,P_j\} = 0 \text{ and } \{Q_i,P_j\} = \delta_{i,j}$
- **Poisson bracket:** $\{f,g\} = \{f,g\}_{\mathbf{q},\mathbf{p}} = \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial p_k}\right)$ ⟶ $\dot{q}_k = \{q_k,H\}$, $\dot{p}_k = \{p_k,H\}$

- Suppose we have a canonical transformation $\mathcal{M}$ such that $H' = H'(\boldsymbol{P})$ (independent of **Q**)

$$\Longrightarrow \quad \dot{P}_k = -\frac{\partial H'}{\partial Q_k} = 0 \Rightarrow \boxed{P_k = \text{constant!}} \quad \textbf{and} \quad \dot{Q}_k = \frac{\partial H'}{\partial P_k} = \text{const.} \Rightarrow \boxed{Q_k(t) = Q_k(0) + \frac{\partial H'}{\partial P_k}t}$$

- In this coordinates, the system is **easily solvable** ⟶ We look for a transformation such that $H\left(\mathbf{q},\frac{\partial M}{\partial \mathbf{q}}\right) = H'(\mathbf{P})$
  - $\mathcal{M}$ has a generating function $M(\boldsymbol{q},\boldsymbol{P})$ such that $p_k = \partial M/\partial q_k$ and $Q_k = \partial M/\partial P_k$

Hamilton-Jacobi equation
(one nonlinear partial differential equation)

Integrable systems

- Hamilton-Jacobi equation is hard to solve in general (except $n = 1$, or separable systems)
- In some cases, this can be achieved via identifying **constants of motion**

Integrability: A Hamiltonian system with n degrees of freedom and $s = n$ **constants of motion** $\{c_i(\boldsymbol{q}, \boldsymbol{p})\}$ is called **integrable**

$\longrightarrow$ **constants** $c_i(\mathbf{q}(t), \mathbf{p}(t)) = c_i(\mathbf{q}(0), \mathbf{p}(0))$

$\longrightarrow$ **involution** $\{c_i, c_j\} = 0 \; \forall \, i, j = 1, \ldots, n$

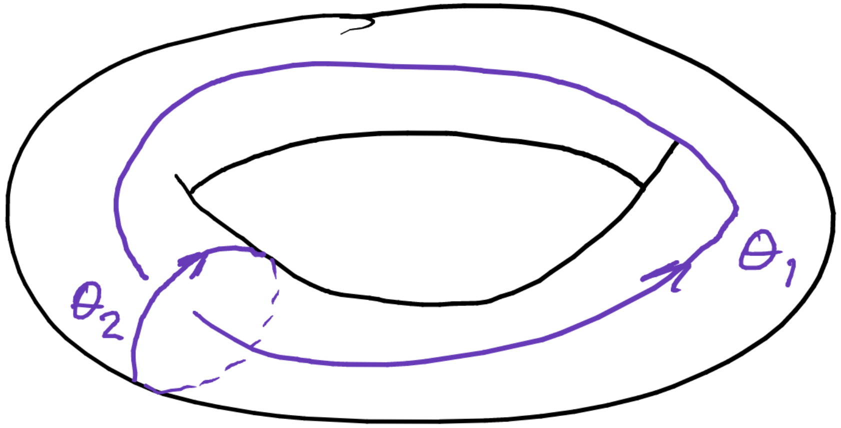
$\longrightarrow$ **independent** vectors ∇c_i are l.i.

- In that case, the canonical transformation $(\mathbf{q}, \mathbf{p}) \longrightarrow (\boldsymbol{\Theta}, \mathbf{I})$ gives a solution to the H-J equation with $\boldsymbol{I_i} = \boldsymbol{c_i}$
- Dynamics of an integrable system is restricted to a **n-dimensional** hypersurface in the 2n-dimensional phase space, which can be thought of as **torus**

$$\theta_k(t) = \theta_k(0) + \omega_k(\mathbf{I})t \longleftarrow$$

$$\text{where } \omega_k(\mathbf{I}) = \frac{\partial H'}{\partial I_k} = \text{const.}$$

Motion is periodic if $\boldsymbol{\omega}.\boldsymbol{k} = 0$ for some $\boldsymbol{k} \in \mathbb{Z}^n$



- *Property*: For $n = 1$, all autonomous Hamiltonian systems are integrable: the required constant of motion is the energy $H(q, p)$

KAM theorem

Let $H = H_0 + \varepsilon H_1$

Integrable (θ, I) — H_0

Non-integrable perturbation — εH_1

How stable are the 'regular' structures (invariant tori) of H_0?

- Kolmogorov – Arnold – Moser (KAM) theorem: for small enough ε, there exists a torus for H with action angle variables

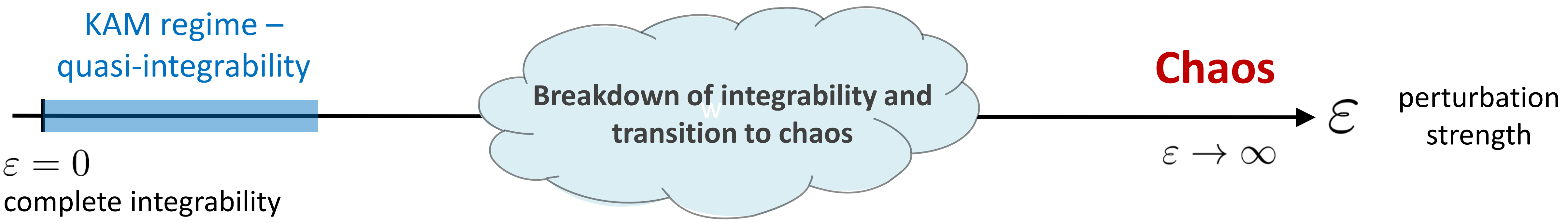
$$\begin{cases} \varphi = \theta + \varepsilon g(\varepsilon, \theta) \\ J = I + \varepsilon f(\varepsilon, \theta) \end{cases}$$

New torus is 'near' the old one, and they transform smoothly

Comment: for a given energy, a tori is fixed, and KAM assumes is nonresonant (motion is not periodic)

Sketch of proof in Section 3.7.4 of "Nonlinear dynamics and quantum chaos: an Introduction" by S. Wimberger

Consequence: regular structures persist under the presence of a small (non-integrable) perturbation. In this regime, the system is 'quasi-integrable' (constant of motion are only approximate)

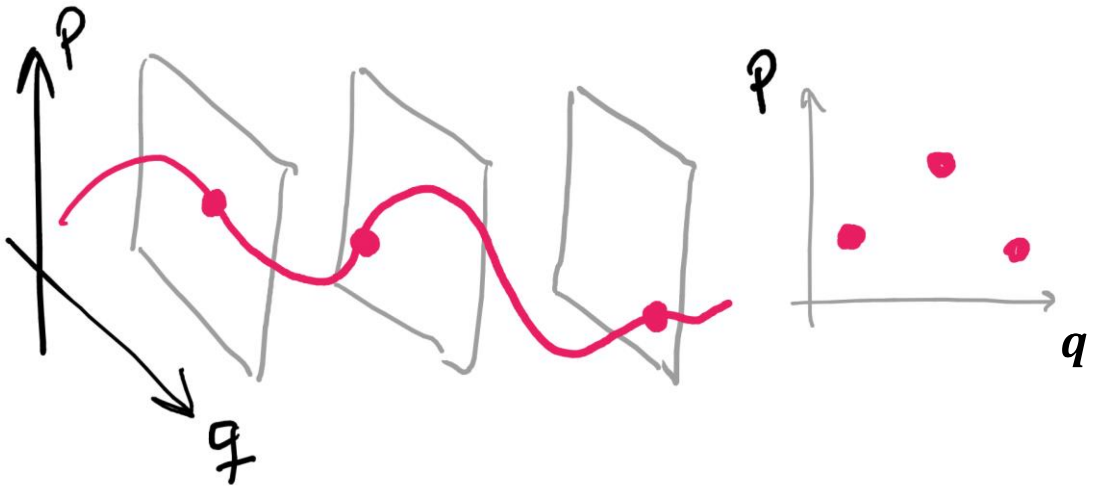


Poincaré maps and transition to chaos

- We want to study the transition between regular and chaotic motion in simple cases: small n (degrees of freedom)
- Recall all autonomous systems with $n = 1$ are integrable – we need to add time-dependence

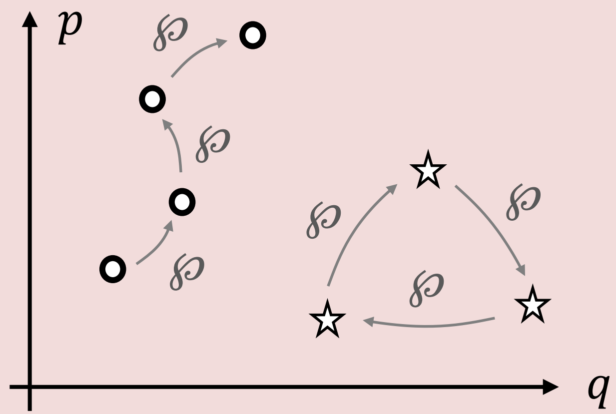
Surface of section (SOS): a tool for visualizing trajectories in low dimensional systems and to identify differences between regular and chaotic motion

- **n=1,** non conservative and periodic $\mathrm{SOS} = \{(q, p, t) : t = nT,\ n \in \mathbb{N}\}$
- **n=2**, conservative $\mathrm{SOS} = \{(\mathbf{q}, \mathbf{p}) : q_2 = 0, H(\mathbf{q},\ \mathbf{p}) = E\}$



Poincaré Map ($\wp$): Gives the evolution in the S.O.S. $\mathbf{z_{n+1}} = \wp(\boldsymbol{z_n})$

- Discrete dynamical system
- Time evolution is unique (each initial condition has its own trajectory)
- Inherits many properties from the continuous dynamical system.
  - Area preserving map (as Liouville's theorem)
  - If the system shows a periodic orbit for an initial condition $z_0 \in SOS$, then the map has a *fixed point* of some order k at $\boldsymbol{z_0}$: $\wp^k(\boldsymbol{z_0}) = \boldsymbol{z_0}$



Fixed points, stability and instability

Fixed points: $\wp(\boldsymbol{z}^*) = \boldsymbol{z}^*$ for $z^* \in SOS$

$\Downarrow$

Near a fixed point, dynamics is given by the **tangent map** $\mathcal{M}(\boldsymbol{z}) = \dfrac{\partial \wp}{\partial \boldsymbol{z}} \overset{n=1}{=} \begin{pmatrix} \dfrac{\partial \wp_q}{\partial q} & \dfrac{\partial \wp_q}{\partial p} \\ \dfrac{\partial \wp_p}{\partial q} & \dfrac{\partial \wp_p}{\partial p} \end{pmatrix}$

$$\wp(\mathbf{z}^* + \delta\mathbf{z}) = \mathbf{z}^* + \mathcal{M}(\mathbf{z}^*).\delta\mathbf{z}$$

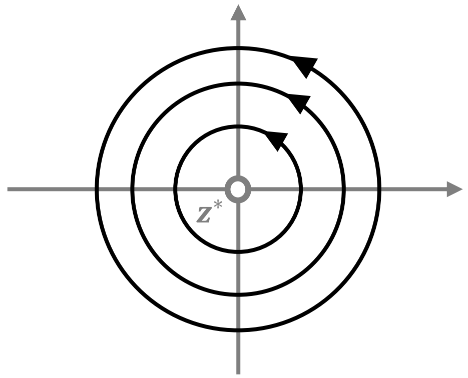
Eigenvalues of $\mathcal{M}(\boldsymbol{z}^*)$:

solutions of $x^2 - Tr(\mathcal{M})\, x + \underbrace{\det(\mathcal{M})}_{=1} = 0$

So, $\mathrm{Tr}(\mathcal{M}(\boldsymbol{z}^*))$ determines type of motion near fixed point:

$|\mathrm{Tr}(\mathcal{M})| < 2 \rightarrow x_\pm = e^{\pm i\beta}$

Elliptic fixed point

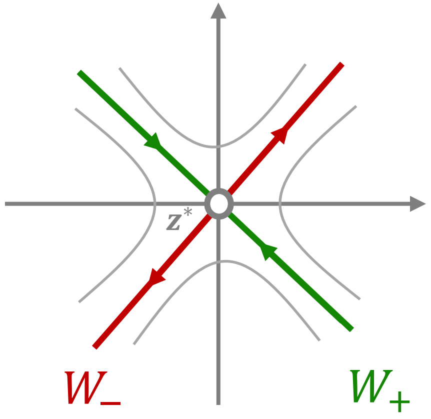


$|\mathrm{Tr}(\mathcal{M})| = 2 \rightarrow x_\pm = \pm 1$

Parabolic fixed point

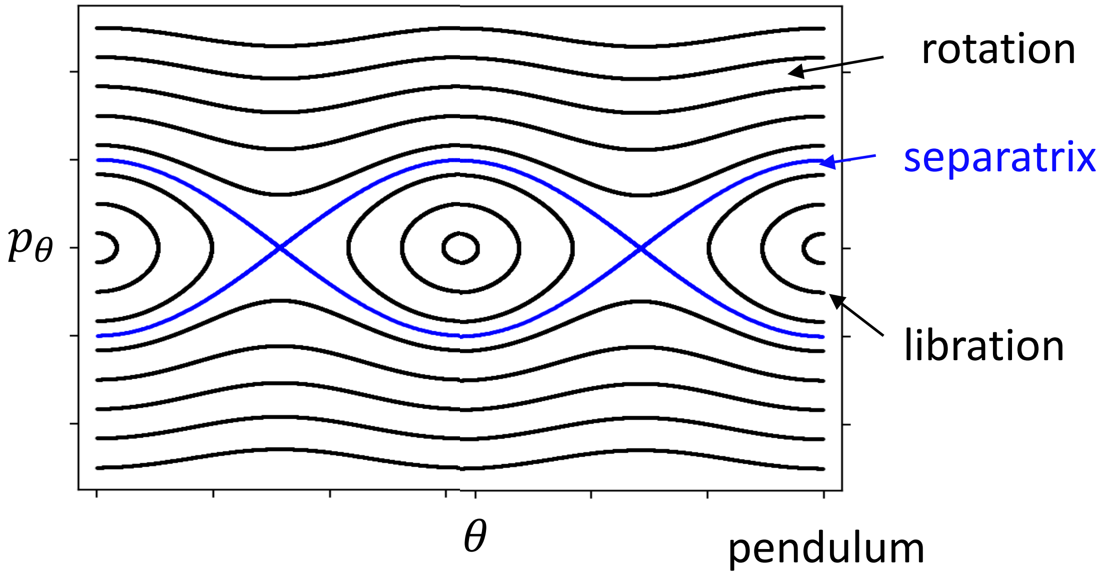
$|\mathrm{Tr}(\mathcal{M})| > 2 \rightarrow x_\pm = e^{\pm\lambda}$

Hyperbolic fixed point



unstable and **stable** manifolds

Separatrix: W_- and W_+ coincide $\Rightarrow$ Very unstable under perturbations!



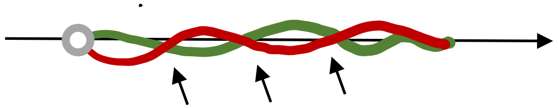
Perturbed motion: all hell breaking loose

We introduce a nonintegrable perturbation: $H = H_0 + \varepsilon H_1$

$\varepsilon = 0 \longrightarrow$ W_- and W_+ **coincide** (separatrix)

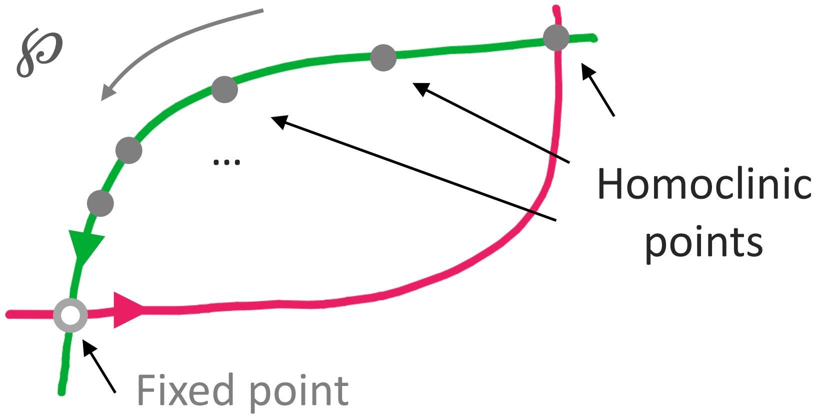
Fixed I

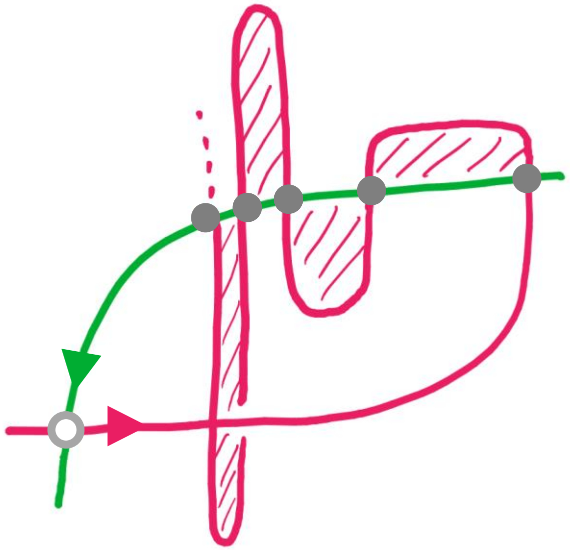
$\varepsilon \neq 0 \longrightarrow$ W_- and W_+ **intersect** (*homoclinic points*)



Variable I (broken integrability)

- A *homoclinic point (HP)* is a point in the SOS which belongs to both W_- and W_+
- Since $W_\pm$ are 'invariant curves' $\wp(W_\pm) = W_\pm$, then if z is a HP $\Rightarrow$ $\wp(z)$ is a HP. So: there is an infinite sequence of them, bunched up together
- Points in the either $W_\pm$ which are not HP's, also evolve according to $\wp$, *area preserving* map

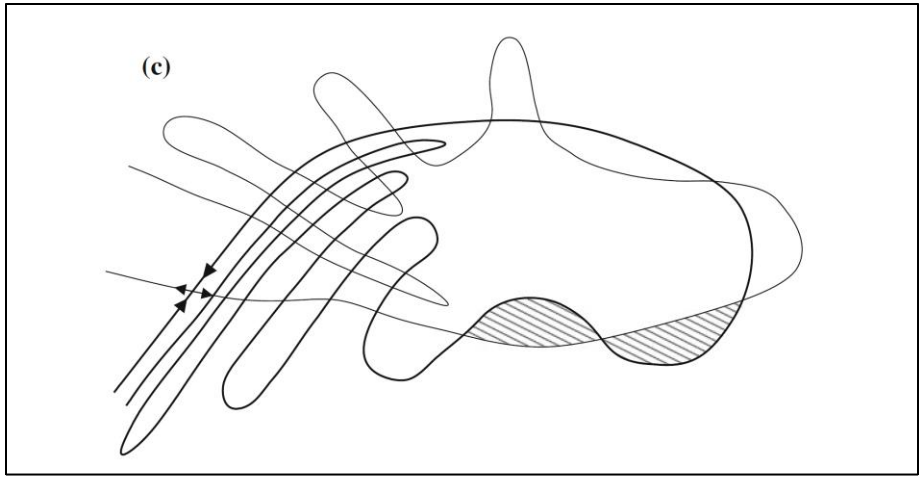




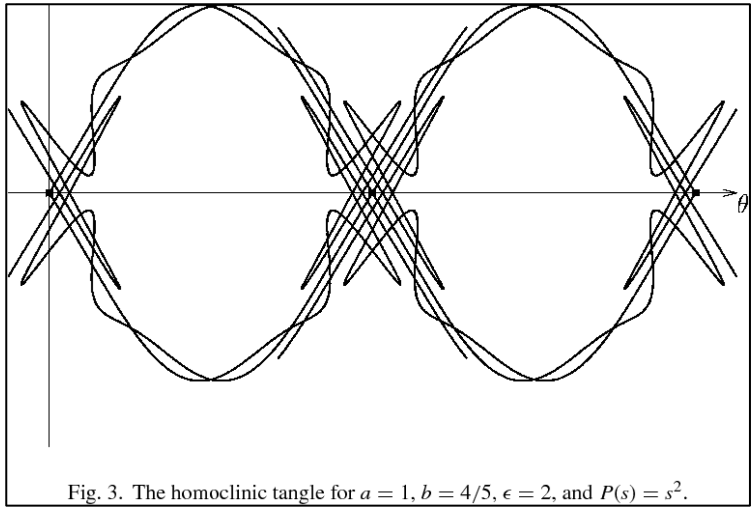
Complicated motion, highly unstable with large deviations with respect to initial conditions! – onset of Chaos

Homoclinic tangle and stochastic layer

Homoclinic tangle



From S. Wimberger, *Nonlinear dynamics and quantum chaos: an Introduction*

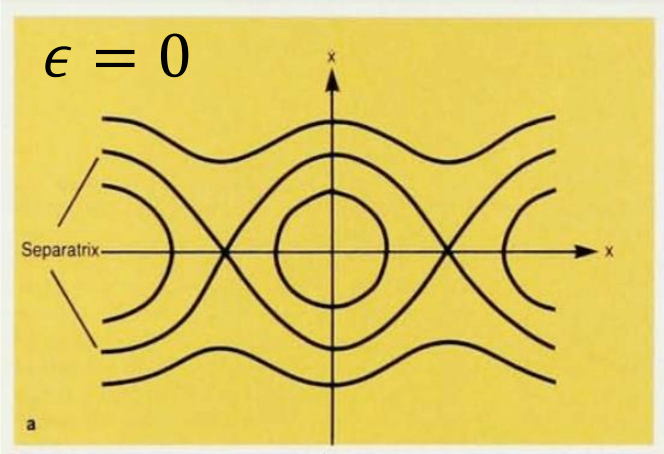


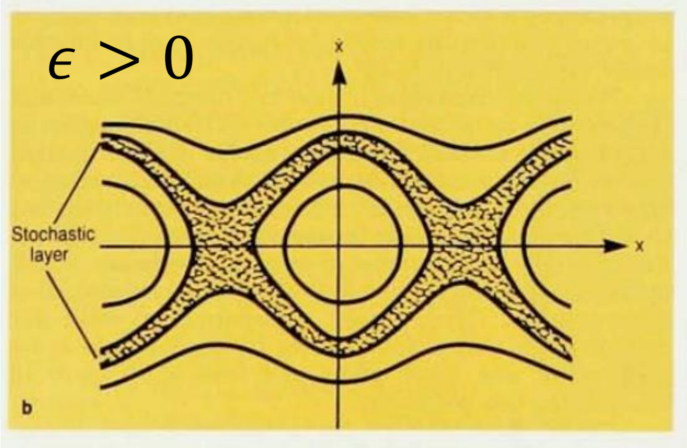
From R. Ramirez-Ros, *Physica D: Nonlinear Phenomena*, ***210*** *149-179 (2005)*

Stochastic layer: complex motion around the (former) separatrix, makes the SOS trajectories look like randomly distributed points

Driven pendulum

$$\ddot{x} + \omega^2 \sin(x) = \epsilon \sin(kx - \Omega t)$$





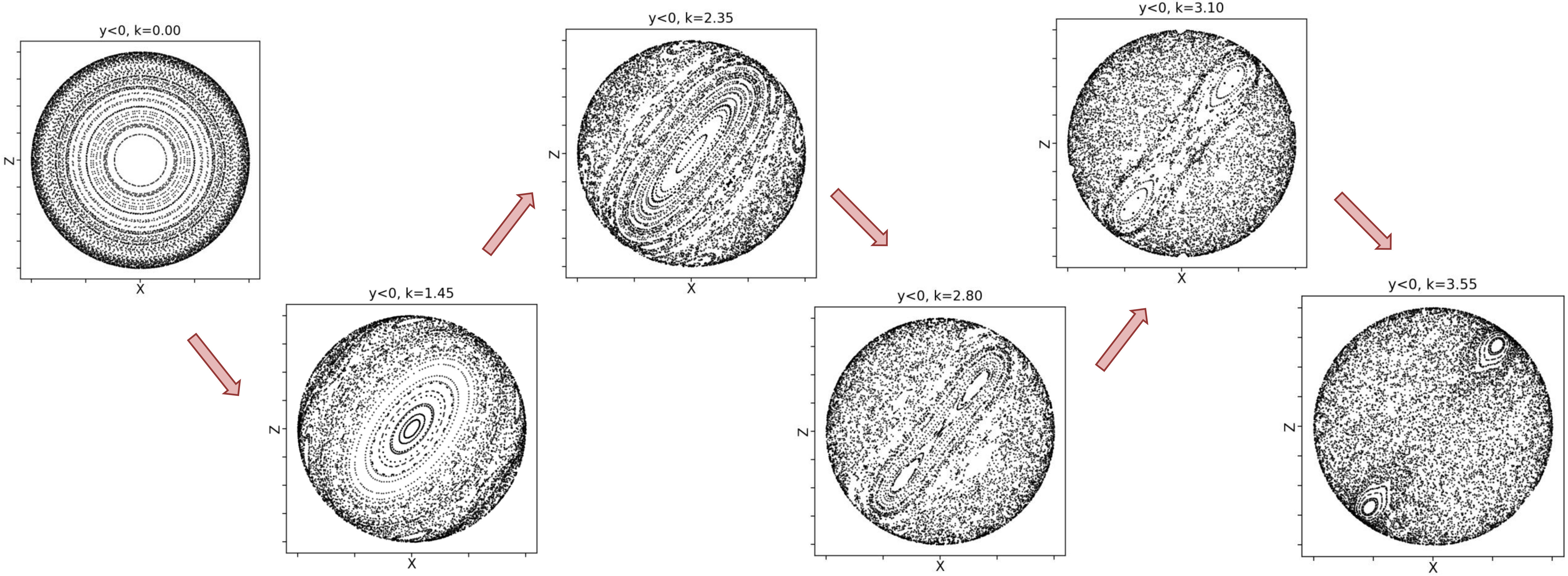
A. Chernikov, R. Zagdeev and G. Zaslavsky, "Chaos: how regular can it be?" Physics Today **41**, 11, 27 (1988)

Transition to chaos in a kicked top

$$H = \underbrace{\frac{\alpha}{\tau} J_y}_{H_0} + \underbrace{\frac{k}{2I} f(t) J_z^2}_{\text{perturbation}} \qquad f(t) = \sum_{m=-\infty}^{m=+\infty} \delta(t - m\tau)$$

Phase space variables are the components of the angular momentum $\boldsymbol{J} = (J_x, J_y, J_z)$. Also, $\boldsymbol{J}^2 = J_x^2 + J_y^2 + J_z^2$ is conserved

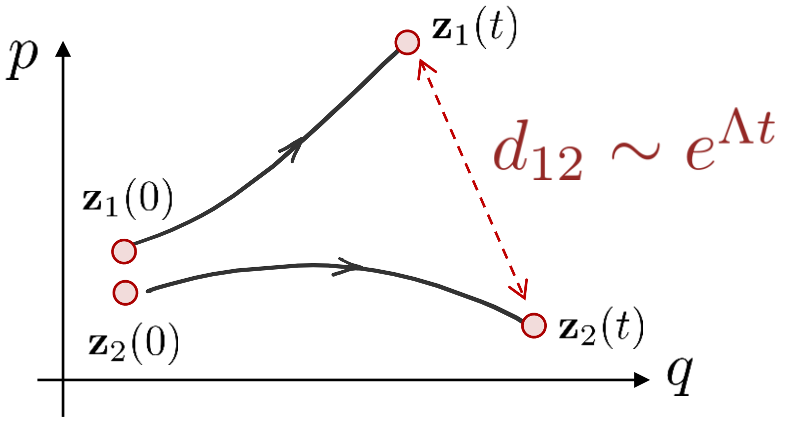
$\Longrightarrow$ **Phase space dimension = 2 $\Rightarrow n = 1$**



F. Haake, M. Kus and R. Scharf, *Classical and quantum chaos for a kicked top*. Z. Phys. B – Cond. Mat. **65**, 381-395 (1987)

Lyapunov exponents

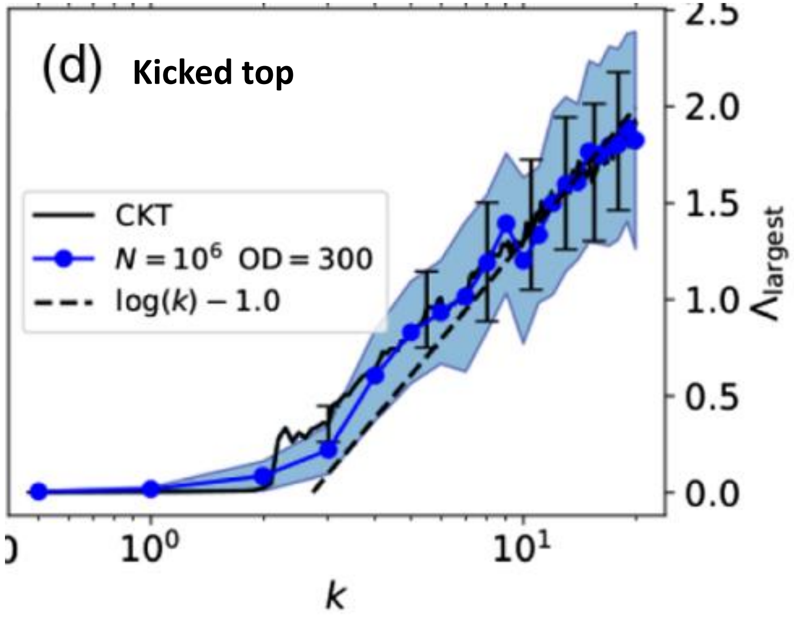
Chaos $\Longrightarrow$ exponential separation of nearby phase space trajectories



- Dynamics near a fixed point $\wp(\mathbf{z}^* + \delta\mathbf{z}) = \mathbf{z}^* + \mathcal{M}(\mathbf{z}^*).\,\delta\mathbf{z}$
- **Hyperbolic** FP: $\mathcal{M}(\mathbf{z}^*) = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix}$ (in normal coordinates)
- After n time steps: $\wp_n = \mathbf{z}^* + \underbrace{e^{n\lambda}\delta\mathbf{z}_u + e^{-n\lambda}}_{\text{Lyapunov exponent }(\lambda)}\delta\mathbf{z}_s$

More generally:

- Maximal Lyapunov exponent $\Lambda(\mathbf{z}) = \lim_{n\to\infty} \lim_{\delta\mathbf{z}\to 0} \frac{1}{n} \log\left(\frac{||\wp_n(\mathbf{z}+\delta\mathbf{z}) - \wp_n(\mathbf{z})||}{||\delta\mathbf{z}||}\right)$
  - For a generic $\delta\mathbf{z}$, $\Lambda(\mathbf{z})$ = max. eigenvalue of $\mathcal{M}(\mathbf{z})$
  - If I take a set of $\{\delta\mathbf{z}_i\}$, I get a **'Lyapunov spectrum'** $\{\lambda_i\}$
- In a globally chaotic regime, $\Lambda(\mathbf{z})$ is mostly independent of initial condition



From M. Muñoz et al PRLPhys. Rev. Lett. **124**, 110503 (2020)

Ergodicity and mixing

Some notation: T: Dynamical system $T: G \times \Omega \to \Omega \longrightarrow T^t(\omega) \in \Omega$ $\quad G \sim$ time $\quad \Omega \sim$ phase space

ν: Measure such that $\nu(\Omega) = 1$, invariant under T: $\nu\left(\mathrm{T}^t(\mathrm{A})\right) = \nu(\mathrm{A})$

Ergodicity **A dynamical system is ergodic if all T-invariant sets ($T^t(A) = A$) are such that either $\nu(A) = 1$ *or* 0**

→ i.e, there cannot be an invariant set which is not a fixed point, or the whole space

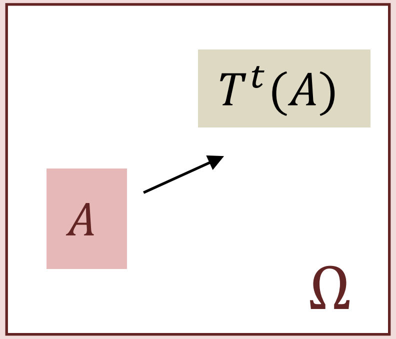
Phase space average = time-average: $\bar{f} = \int_\Omega f \, d\nu$ where $\bar{f}(\omega) = \lim_{t\to\infty} \sum_{n=0}^{t} f\left(T^n(\omega)\right)$

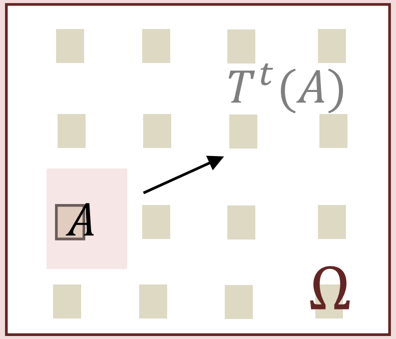
Orbits fill the whole available phase space

Note that: **Ergodicity** $\not\Rightarrow$ **Chaos** → Integrable systems can lead to ergodicity in the available phase space (torus)

Mixing **A dynamical system is (strongly) mixing if for any two sets $A, B \subseteq \Omega$,** $\lim_{t\to\infty} \nu\left(A \cap T^t(B)\right) = \nu(A)\nu(B)$

Not mixing





Mixing

Mixing $\Rightarrow$ Ergodicity $\quad$ **Chaos** $\Rightarrow$ Mixing

Ergodic hierarchy and correlation functions

Take a **correlation function** $C(t; f, g) = \langle f\left(T^t(x)\right) g(x)\rangle - \langle f(x)\rangle\langle g(x)\rangle$

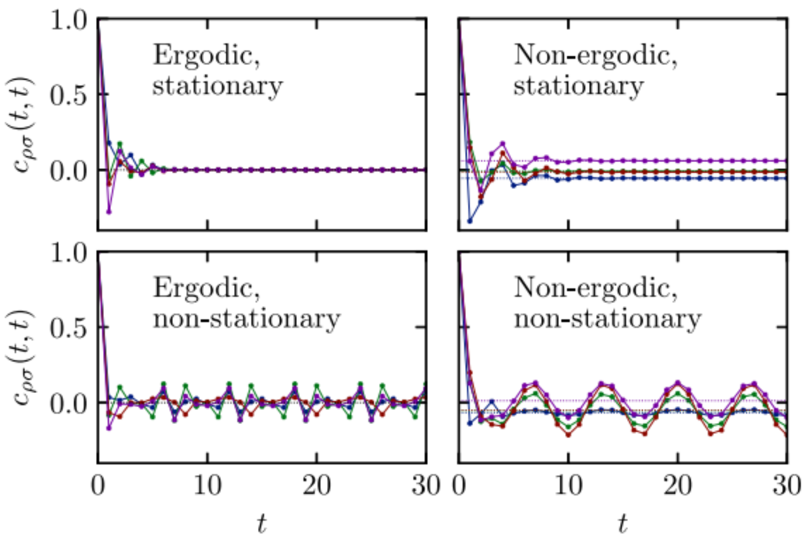
with $f(x),\ g(x)$ functions in Ω

and $\langle h(x)\rangle = \int_\Omega h(x)\, d\nu$ $\qquad \bar{f}(\omega) = \lim_{t\to\infty} \sum_{n=0}^{t} f\left(T^n(\omega)\right)$

Ergodic hierarchy

Ergodic		**Weak Mixing**		**Mixing**		
$\overline{C(t)} = 0$	$\supset$	$\overline{\lvert C(t)\rvert} = 0$	$\supset$	$C(t) \to 0$	$\supset$	…

- Analyzing the behavior of correlation functions is useful for systems with many d.o.f.s where is hard to characterize chaos 'globally' (Lyapunovs, etc)



From P. Claeys and A. Lamacraft, "Ergodic and Nonergodic Dual-Unitary Quantum Circuits with Arbitrary Local Hilbert Space Dimension" PRL **126** 100603 (2021)

Summary

- Integrable systems are characterized by conserved quantities which allow for the dynamics to be solved using action-angle variables. Motion lies on *invariant tori*
- Upon addition of a nonintegrable perturbation, tori persist (KAM theorem) for a while. After that, proliferation of instabilities leads to complex motion, and hypersensitivity to initial conditions
- Global chaotic behavior can be characterized at short times via Lyapunov exponents, and for long times via mixing and ergodicity

Next week: Introduction to quantum chaos. Some good reads to get some background:

- F. Haake – 'Quantum signatures of chaos' – Chapter 1
- M. Berry – 'Chaos and the semiclassical limit of quantum mechanics' (link)
- D. Poulin – 'A rough guide to quantum chaos' (link)

References

- Main reference: *Nonlinear dynamics and quantum chaos: an Introduction* – by Sandro Wimberger
- S. Aravinda et al, *From dual-unitary to quantum Bernoulli circuits: Role of the entangling power in constructing a quantum ergodic hierarchy*, arxiv:2101.04580.
- A. Chernikov, R. Zagdeev and G. Zaslavsky, *Chaos: how regular can it be?.* Physics Today **41**, 11, 27 (1988)
- Notes from the course "Chaos and Quantum Chaos 2021" at TU Dresden, taught by Roland Ketzmerick. Some materials publicly available in English at https://tu-dresden.de/mn/physik/itp/cp/studium/lehrveranstaltungen/chaos-and-quantum-chaos-2021?set_language=en

Extra stuff

Autonomous and non autonomous systems

- $H(q,p)$ time-independent $\Rightarrow H\big(q(t),p(t)\big) = H\big(q(0),p(0)\big) = E$
- $H(q,p,t) = E(t)$ explicitly time-dependent

New EOMs

Can be mapped to an autonomous system in an extended phase space

$$\underset{\substack{\uparrow\\ q_0}}{\overset{\substack{p_0\\ \downarrow}}{\begin{array}{c} p' = (-E,p) \\ q' = (t,q) \end{array}}} \Longrightarrow \mathcal{H}(p',q') = H(q,p,t) - E(t) \Longrightarrow \frac{dq'}{d\tau} = \frac{\partial\mathcal{H}}{\partial p\prime}, \frac{dp'}{d\tau} = -\frac{\partial\mathcal{H}}{\partial q'}$$

New Hamiltonian

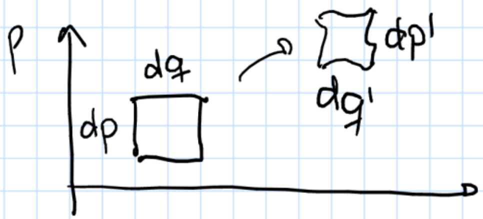
This leads to

- $\frac{dq_0}{d\tau} = \frac{\partial\mathcal{H}}{\partial p_0} \leftrightarrow \frac{dt}{d\tau} = -\frac{\partial\mathcal{H}}{\partial E} = 1 \ \rightarrow \tau = t + const.$ ⟵ This variable becomes ‘trivial’
- $\frac{dp_0}{d\tau} = -\frac{\partial\mathcal{H}}{\partial q_0} \leftrightarrow -\frac{dE}{dt} = -\frac{\partial\mathcal{H}}{\partial t}$ ⟵ One extra independent variable

“one and a half degrees of freedom”

Also $\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}$ for the old variables

Extra: Liouville's Theorem for $n = 1$



$$\begin{cases} q' = q + \dot{q}\,\delta t \\ p' = p + \dot{p}\,\delta t \end{cases}$$

Then $dq' = dq + d\dot{q}\,\delta t = dq + \frac{d\dot{q}}{dq}\,dq\,\delta t$

$dp' = dp + \frac{d\dot{p}}{dp}\,dp\,\delta t$

$$\Rightarrow dq'dp' = dq\,dp\left[1 + \delta t\left(\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p}\right)\right]$$

Note: $\dot{q} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial q}$

$$\Rightarrow dq'dp' = dq\,dp\left[1 + \delta t\underbrace{\left(\frac{\partial H}{\partial q\,\partial p} - \frac{\partial H}{\partial p\,\partial q}\right)}_{0}\right]$$

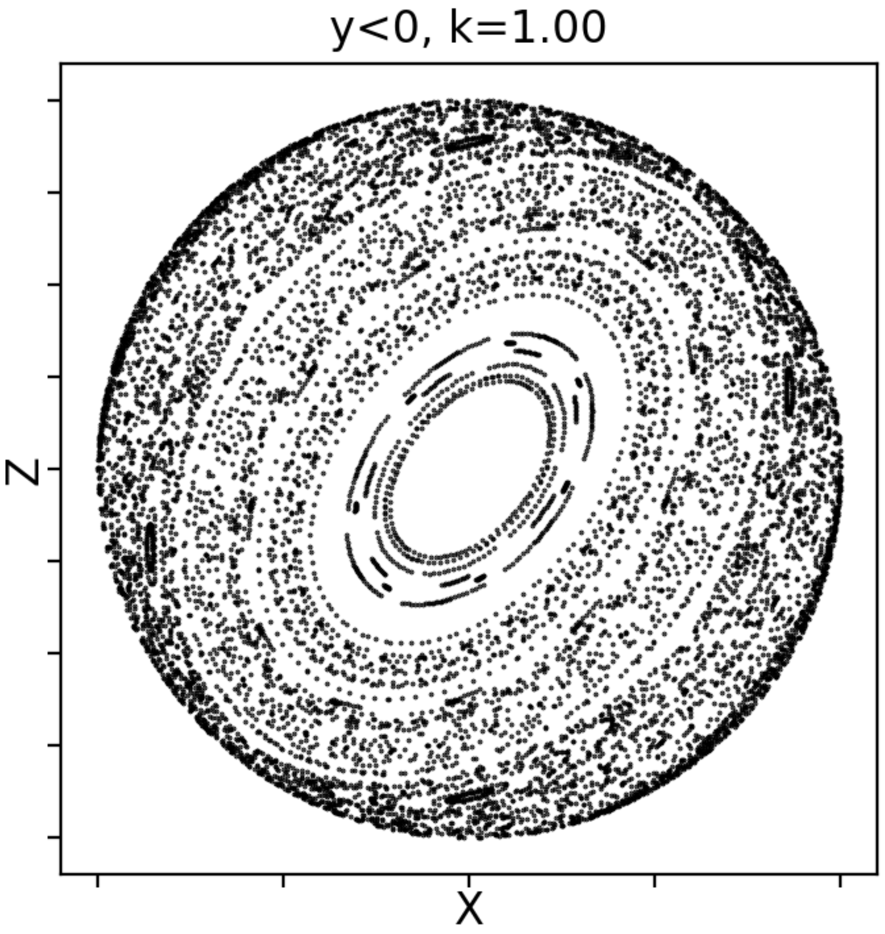
$$\Rightarrow dq'dp' = dq\,dp$$

$$H = \underbrace{\frac{\alpha}{\tau} J_y}_{H_0} + \underbrace{\frac{k}{2I} f(t) J_z^2}_{\text{perturbation}}$$

$$f(t) = \sum_{m=-\infty}^{m=+\infty} \delta(t - m\tau)$$

Phase space variables are the components of the angular momentum $\boldsymbol{J} = (J_x, J_y, J_z)$. Also, $\boldsymbol{J}^2 = J_x^2 + J_y^2 + J_z^2$ is conserved

$\Longrightarrow$ **Phase space dimension = 2 $\Rightarrow$ $\boldsymbol{n} = 1$**



Existence / separability of H-J equations and integrability

Let there be given a $2n$-dimenional real symplectic manifold (M, ω) with a globally defined real function $H : M \times [t_i, t_f] \to \mathbb{R}$, which we will call the *Hamiltonian*. The time evolution is governed by Hamilton's (or equivalently Liouville's) equations of motion. Here $t \in [t_i, t_f]$ is time.

1. On one hand, there is the notion of *complete integrability*, aka. *Liouville integrability*, or sometimes just called *integrability*. This means that there exist n independent **globally** defined real functions

   $$I_i, \qquad i \in \{1, \ldots, n\},$$

   (which we will call *action variables*), that pairwise Poisson commute,

   $$\{I_i, I_j\}_{PB} = 0, \qquad i, j \in \{1, \ldots, n\}.$$

2. On the other hand, given a fixed point $x_{(0)} \in M$, under mild regularity assumptions, there always exists **locally** (in a sufficiently small open Darboux[1] neighborhood of $x_{(0)}$) an n-parameter complete solution for *Hamilton's principal function*

   $$S(q^1, \ldots, q^n; I_1, \ldots, I_n; t)$$

   to the Hamilton-Jacobi equation, where

   $$I_i, \qquad i \in \{1, \ldots, n\},$$

   are integration constants. This leads to a **local** version of property 1.

The main point is that the **global** property 1 is rare, while the **local** property 2 is generic.

From: 'Integrable vs non-integrable systems' on Physics.StackExchange. See also: 'Constants of motion vs integrals of motion vs first integrals' on Physics.StackExchange