Summer study course on many-body quantum chaos

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QUANTUM SYSTEMS ACCELERATOR
Catalyzing the Quantum Ecosystem

Q-SEnSE
An NSF Quantum Leap Challenge Institute

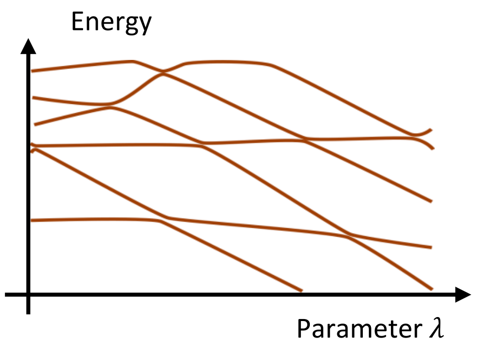
Session 2: Quantum chaos in systems with few degrees of freedom

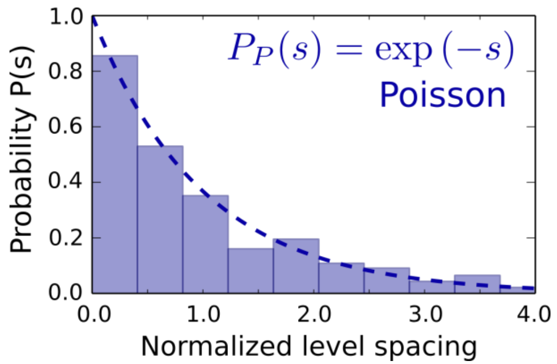
Wednesday June 9th 2021

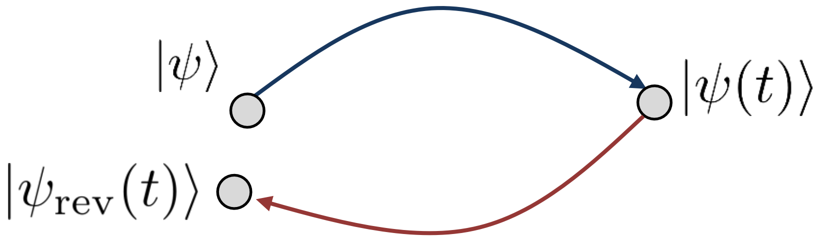
Background picture: Ghost Ranch, New Mexico (USA)

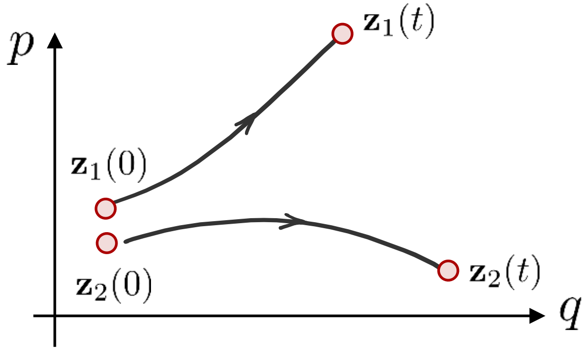
Today's menu

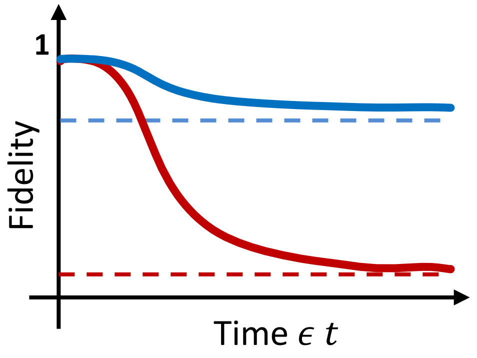
1. **Signatures of chaos in the energy spectrum**
   1. **Level repulsion**
   2. **Level spacing statistics**
2. **Signatures of chaos in the energy eigenstates**
3. **Signatures of chaos in in quantum dynamics**
   1. **Ehrenfest time**
   2. **Loschmidt echo**









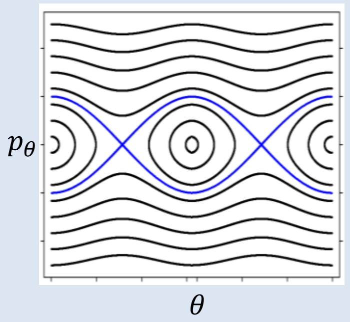


Chaos and integrability in the energy spectrum

Classical

perturbation

Integrable

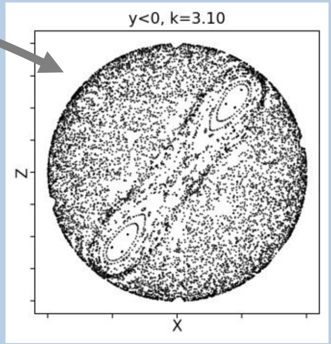


n constants of motion

solvable H-J equation

motion on a n-dim. torus

Chaotic

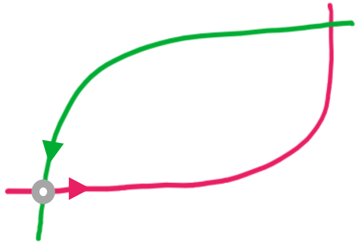


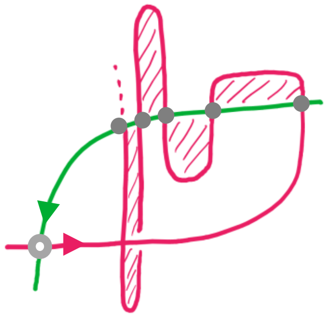
motion in a $2n$-dim. phase space

instabilities and sensitivity to initial conditions

mixing and ergodicity

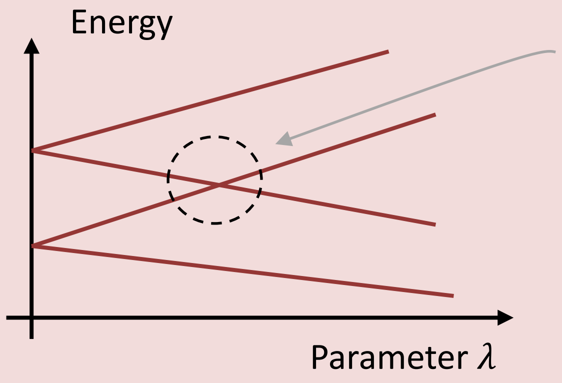
Constraints
Less generic

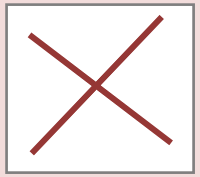




No constraints
More generic

Quantum



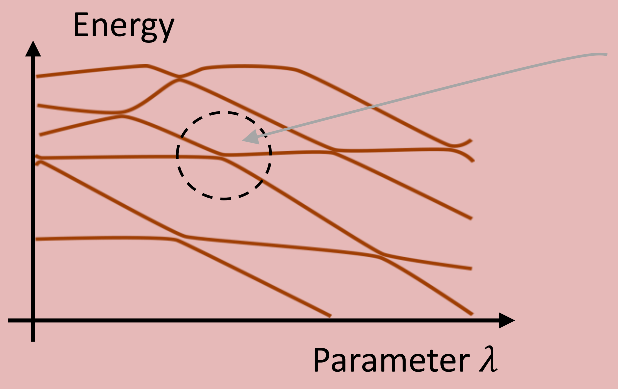


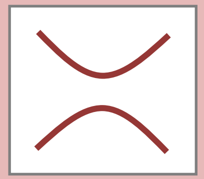
Exact level crossing

Level clustering

⇕

Quantum 'regular'





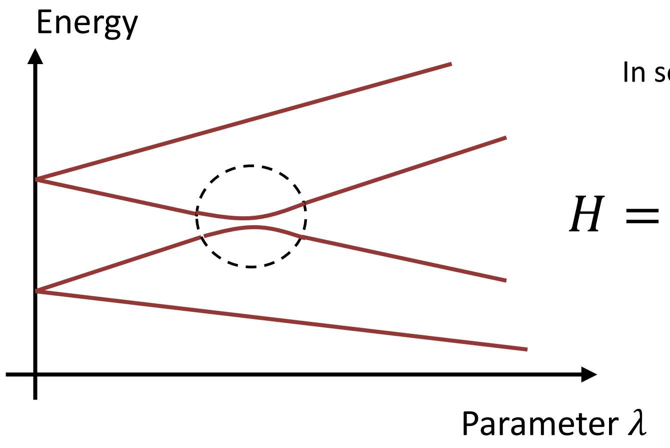
***Avoided* level crossing**

Level repulsion

⇕

Quantum 'chaotic'

Model for level dynamics



In some basis $\{|1\rangle, |2\rangle\}$

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}$$

Eigenvalues: $E_\pm = \frac{1}{2}(H_{11} + H_{22}) \pm \frac{1}{2}\sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2}$

Difference: $\Delta E = E_+ - E_- = \sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2}$

When does $\boldsymbol{\Delta E = 0}$?

- If $H_{12} = 0$, H is a function of two real parameters, $\Delta E = |H_{11} - H_{22}|^2$

  $\Delta E = 0$ by tuning $\boldsymbol{k = 1}$ **parameter**

- If $H_{12} \in \mathbb{R}$, H is a function of three real parameters, $\Delta E = \sqrt{(H_{11} - H_{22})^2 + 4\, H_{12}^2}$

  $\Delta E = 0$ by tuning $\boldsymbol{k = 2}$ **parameters**

- If $H_{12} \in \mathbb{C}$, H is a function of four real parameters, $\Delta E = \sqrt{(H_{11} - H_{22})^2 + 4\, Re(H_{12})^2 + 4\, Im(H_{12})^2}$

  $\Delta E = 0$ by tuning $\boldsymbol{k = 3}$ **parameters**

↳ Level crossing 'codimension' k

Less constraints

⇕

Exact crossing requires tuning more parameters

Increasing 'level repulsion'

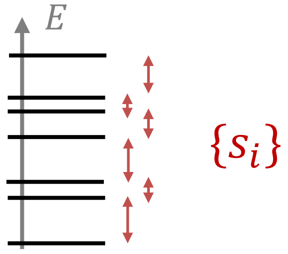
Model for level dynamics

Level repulsion

If H is chosen at random, how likely is it that adjacent levels cross (are degenerate)? $\longrightarrow$

Level spacing distribution $P(s)$

(for small s) $s_i = \Delta E_i = E_{i+1} - E_i$

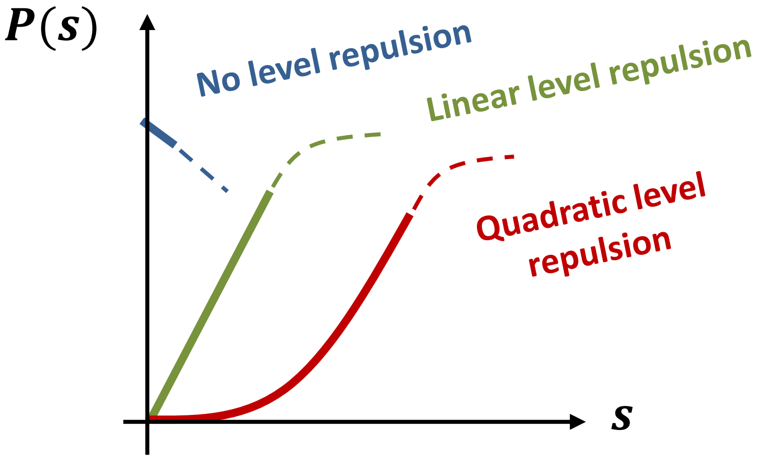


For our 2x2 model:

$$\boldsymbol{P(s)} = \langle \boldsymbol{\delta(s - \Delta E)} \rangle \text{ with } \Delta E = \sqrt{\underbrace{(H_{11} - H_{22})^2}_{x^2} + \underbrace{4\,Re(H_{12})^2}_{y^2} + \underbrace{4\,Im(H_{12})^2}_{z^2}} \quad \Rightarrow P(s) = \int dx\,dy\,dz\,P(x,y,z)\,\delta(s - \mathrm{r})$$

Approximating $P(\vec{r}) \simeq P_o$ constant near $s = 0$

- If $H_{12} = 0\ (k = 1) \Rightarrow P(s) \sim \int dx\,\delta(s - \mathrm{x}) \sim const$ (independent of s)
- If $H_{12} \in \mathbb{R}\ (k = 2) \Rightarrow P(s) \sim \int dx \int dy\,\delta(s - r) \sim 2\pi \int dr\,r\,\delta(s - r) \sim \boldsymbol{s}$
- If $H_{12} \in \mathbb{C},\ (k = 3) \Rightarrow P(s) \sim \int dx \int dy \int dz\,\delta(s - r) \sim 4\pi \int dr\,r^2\,\delta(s - r) \sim \boldsymbol{s^2}$



Level repulsion implies **correlation** of the energy levels. In absence of level repulsion, the **levels are uncorrelated** with each other

Level clustering in integrable systems

For integrable systems, semiclassical methods can be used to compute the energy levels

Einstein-Keller-Brillouin (EKB) quantization: $$I_i = \frac{1}{2\pi}\oint \boldsymbol{p}.d\boldsymbol{q} = \hbar\left(m_i + \frac{\mu_i}{4}\right)$$

$\sim$ Bohr-Sommerfeld

Integrable systems with arbitrary dimension

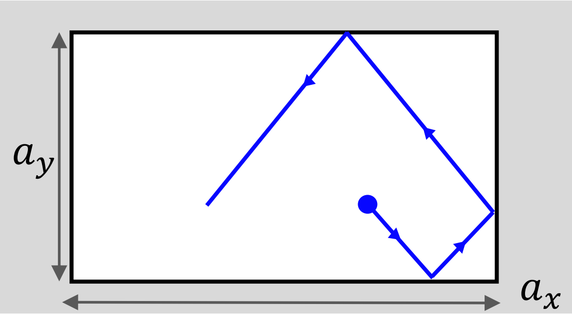
Valid for large action $I \gg \hbar$

$$\Downarrow$$

Classical energy evaluated at discrete actions $\rightarrow$ $\boxed{H'(\boldsymbol{I}_{\vec{\boldsymbol{m}}}) = E_{\{\vec{\boldsymbol{m}}\}}}$ $\leftarrow$ Allowed energies

Energy levels are determined by a set of quantum numbers $\vec{\boldsymbol{m}}$. Levels with completely different m's can have the same energy (typically, if # d.o.f. > 1) - there is *no correlation*

Example: **rectangular Billiard**

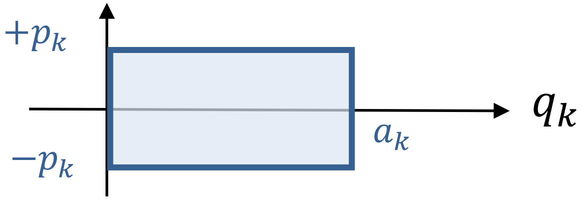


$$V(x,y) = \begin{cases} 0 \ \text{ if } \ 0 \le x \le a_x \\ \qquad\ \ 0 \le y \le a_y \\ \infty \ \text{ otherwise} \end{cases}$$

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) = \frac{\pi^2}{2m}\left(\frac{I_x^2}{a_x^2} + \frac{I_y^2}{a_y^2}\right)$$

constant

Actions: $$I_k = \frac{1}{2\pi}\oint p_k dq_k = \frac{a_k p_k}{\pi}$$



Quantization: $I_k = \hbar(m_k + 1) \equiv \hbar n_k$

$$\Downarrow$$

$$\boxed{E_{n_x,n_y} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{a_x^2} + \frac{n_y^2}{a_y^2}\right)}$$

(particle in a 2D box)

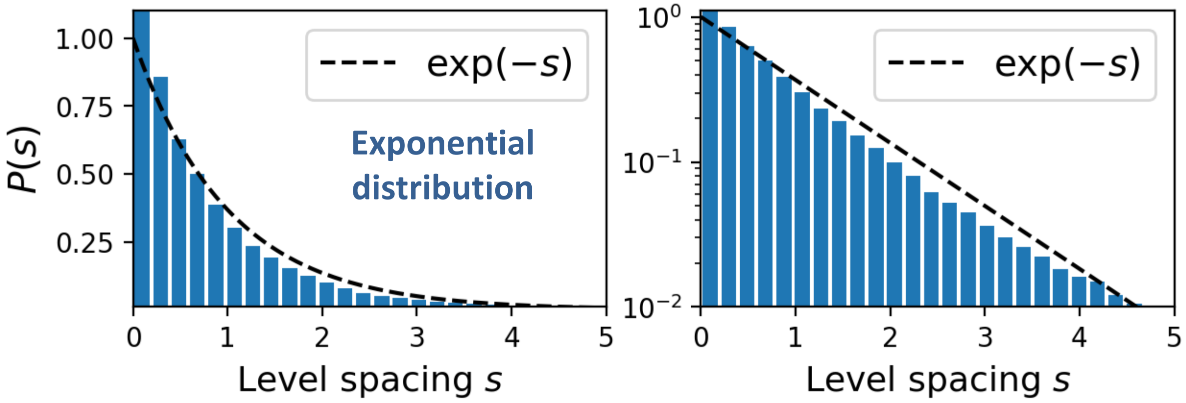
Berry-Tabor conjecture

Rectangular Billiard: $E_{n_x,n_y} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{a_x^2} + \frac{n_y^2}{a_y^2}\right)$

Level spacing distribution $P(s)$

(for incommensurate a_x^2 and a_y^2)

$a_x = 1, a_y = \frac{1}{2}(\sqrt{5} - 1)$

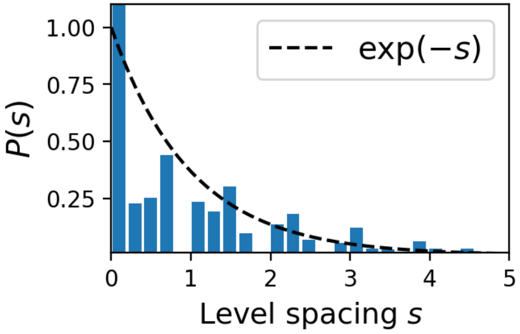


Berry – Tabor (B-T) conjecture: In the limit of large energies (semiclassical limit), the level spacing statistics of the quantum spectra of classically integrable systems correspond to the prediction for *randomly* distributed energy levels, and follow the exponential distribution $P(s) = e^{-s}$

M. V. Berry and M. Tabor, *Level clustering in the regular spectrum.* Proc. R. Soc. London A **356,** 375-394 (1977)

Exceptions

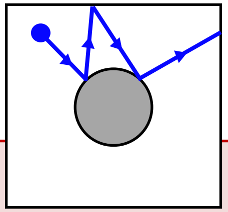
- Systems with one degree of freedom (all of them are integrable anyway)
- Linear systems (quadratic Hamiltonians)
- Systems with closed orbits (commensurate frequencies)



$a_x = 1, a_y = \sqrt{3}$

Level repulsion for nonintegrable systems

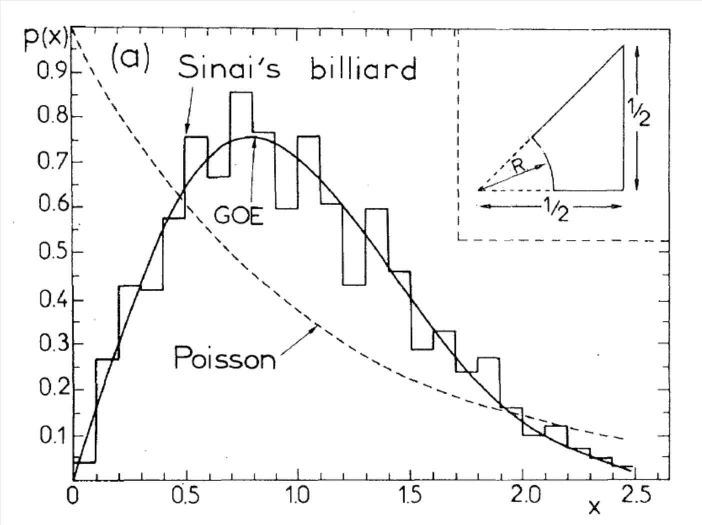
- For nonintegrable systems, the semiclassical methods cannot be used to compute the energies anymore
- Lifting constraints → **Level repulsion**



Bohigas-Giannoni-Schmidt (BGS) conjecture: the eigenvalues of a quantum system whose classical analogue is *fully* chaotic, obey the statistics of level spacing predicted by Random Matrix Theory, and in particular those from the Gaussian random ensembles.

$$P(s) = \begin{cases} \frac{\pi}{2} s \exp(-\frac{\pi}{4} s^2) & \text{GOE: (real, symmetric) random matrices, elements} \sim \text{Gaussian} \\ \frac{32}{\pi} s^2 \exp(-\frac{4}{\pi} s^2) & \text{GUE: (complex, hermitian) random matrices, elements} \sim \text{Gaussian} \end{cases}$$

Wigner-Dyson distributions



Next week (June 16th): Random Matrix Theory!

O. Bohigas, M. J. Giannoni, and C. Schmit, *Characterization of chaotic quantum spectra and universality of level fluctuation laws,* Phys. Rev. Lett. **52** 1, 1-3 (1984)

Extensions to general systems

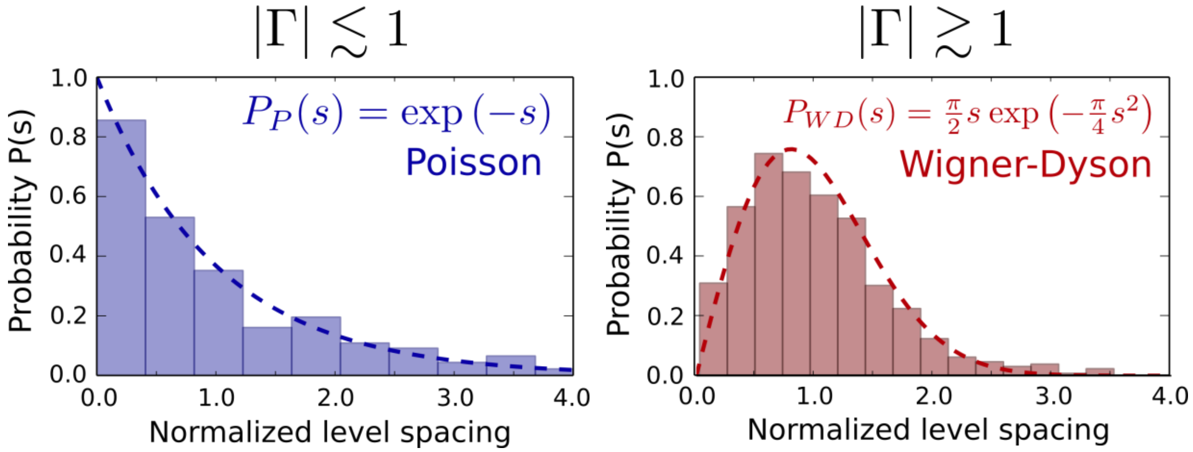
Level spacing statistics is often taken as the *definition* of quantum chaos

Integrable (Bethe ansatz)

$$H_0 = \frac{J}{2}\sum_{i=1}^{L-1} \sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \alpha_z\sigma_i^z\sigma_{i+1}^z \quad \text{(Heisenberg XXZ)}$$

$$H_1 = \frac{J}{2}\sum_{i=1}^{L-2} \sigma_i^x\sigma_{i+2}^x + \sigma_i^y\sigma_{i+2}^y + \alpha_z\sigma_i^z\sigma_{i+2}^z \quad \text{and} \quad \boxed{H_{01} = H_0 + \Gamma H_1}$$

Nonintegrable



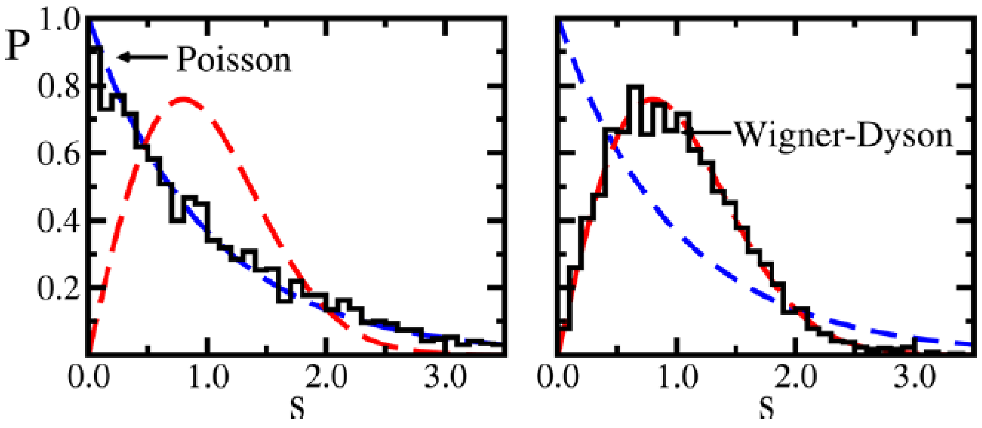
From P. Poggi and D. Wisniacki, Phys. Rev. A **94**, 033406 (2016)
See also L. F. Santos, F. Borgonovi, and F. M. Izrailev, Phys. Rev. E **85**, 036209 (2012).

$$H = H_z + H_{\text{NN}},$$

where

$$H_z = \sum_{i=1}^{L} \omega_i S_i^z = \left(\sum_{i=1}^{L} \omega S_i^z\right) + \epsilon_d S_d^z, \qquad \text{defect}$$

$$H_{\text{NN}} = \sum_{i=1}^{L-1} \left[J_{xy}\left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y\right) + J_z S_i^z S_{i+1}^z\right].$$



Defect at edge

Defect at L/2

From A. Gubin and L. Santos, Am. J. Phys. **80,** 246 (2012)

Eigenstates of integrable systems

- In integrable systems, semiclassical methods can also be used to approximate **eigenstates (WKB theory)**
- Eigenstates tend to localize around the regular structures $W_{\vec{n}}(\mathbf{q},\mathbf{p}) \simeq \delta\left(\mathbf{I}(\mathbf{q},\mathbf{p}) - \mathbf{I}_{\vec{n}}\right) + \mathcal{O}(\hbar)$

Wigner function

Classical action

Quantized action

This is explained in Wimberger's book 4.2.2, 4.2.4 and 4.4

- In chaotic systems, there is no tori, and eigenstates tend to be irregular, and **smeared out** over chaotic regions

Eigenstate delocalization

Inverse participation ratio (IPR): $\eta_{IPR} = \sum_k |\langle \phi_k | \psi \rangle|^4$

- Measures concentration of $|\psi\rangle$ on a basis $\{|\phi_k\rangle\}$
- In phase space, these could be coherent states
- In general settings, other choices are possible, for instance **site basis** or **mean field** basis

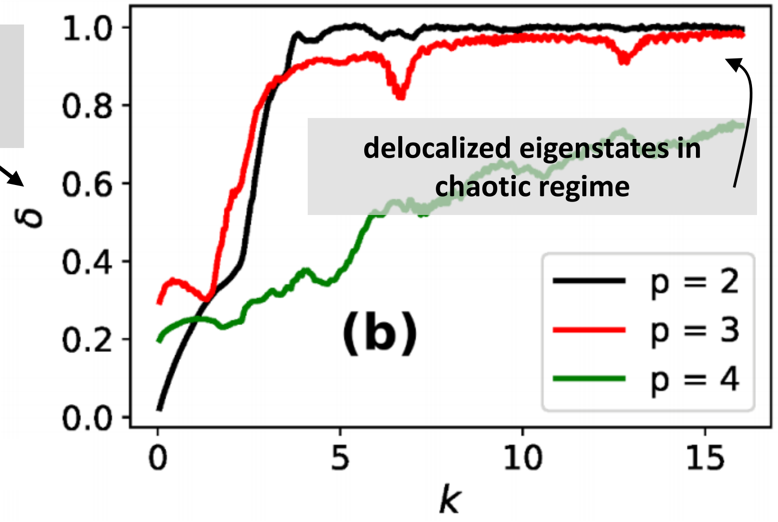
delocalized *localized*

d^{-1} —— 1 → η_{IPR}

$$H = \frac{\alpha}{\tau} J_y + \frac{k}{pJ^{p-1}} f(t) J_z^p \qquad f(t) = \sum_{m=-\infty}^{m=+\infty} \delta(t - m\tau)$$

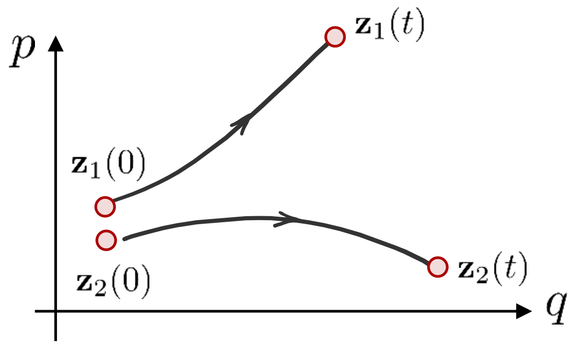
Averaged 'participation ratio' (η_{IPR}^{-1}/d)

Here, $|\psi\rangle$ is an eigenstate of H, and $|\phi_k\rangle$ eigenstate of J_y. Average is over all $|\psi\rangle$



From M. Muñoz et al Phys. Rev. E 103, 052212 (2021)

Ehrenfest time



- Classical chaos → exponential separation of **trajectories** in phase space
- Quantum dynamics → linear evolution of vectors in Hilbert space

What about expectation values? **Ehrenfest theorem**

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \longrightarrow \begin{cases} \dfrac{d\langle q(t)\rangle}{dt} = \dfrac{\langle p(t)\rangle}{m} \\ \dfrac{d\langle p(t)\rangle}{dt} = \langle F(q(t))\rangle \end{cases}$$

where $\langle A(t)\rangle = \langle \psi(t)|\hat{A}|\psi(t)\rangle$

In general

$$\langle F(q(t))\rangle \neq F(\langle q(t)\rangle)$$

One can expand $F(q)$ to obtain $\langle F(q)\rangle = F(\langle q\rangle) + \frac{1}{2}(\Delta q)^2 \left.\frac{d^2F}{dq^2}\right|_{q=\langle q\rangle}$

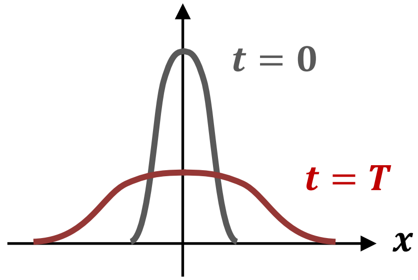
$(\Delta q)^2 = \langle (q - \langle q\rangle)^2\rangle$

As long as the wave packet is localized, its first moments evolve according to Hamilton's equations

(Ehrenfest correspondence)

In time, an initially localized wave packet will **diffuse** → at some point, the correspondence breaks down ⟶ **Ehrenfest time t_E**

i.e. a minimum uncertainty Gaussian wavepacket, free evolution



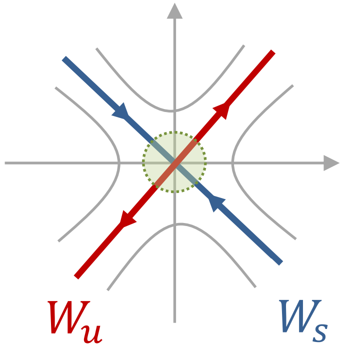
$$\sigma_x^2(T) = \sigma_x^2(0)\left(1 + \left(\frac{T}{\tau}\right)^2\right)$$

action

$$T \sim \frac{\sigma_x(T)}{\sigma_x(0)} \sim \frac{L\,\sigma_p(0)}{\hbar} \sim \frac{S_0}{\hbar}$$

t_E for regular systems scales as $\left(\frac{S_0}{\hbar}\right)^{\alpha}$ → for 'macroscopic' action, these times are very large

Ehrenfest time for chaotic systems



Chaotic systems have exponential instabilities

$$\sigma_u(t) \simeq \sigma_u(0)e^{\lambda t} \Rightarrow t = \lambda^{-1}\log(\sigma_u(t)/\sigma_u(0)) \Rightarrow \boxed{t_E \simeq \lambda^{-1}\log\left(\frac{S_0}{\hbar}\right)}$$

Lyapunov exponent

In chaotic systems, the Ehrenfest correspondence breaks down in a shorter timescale than for regular systems

Hyperion $\longrightarrow$ 'potato-shaped' moon of Saturn with chaotic motion (tumbling), $\lambda = 100\ days^{-1}$

$\longrightarrow$ Berry estimates $\frac{S_0}{\hbar} \sim 10^{58} \rightarrow t_E \sim 100\ days \times \log 10^{58} \sim 37\ years$

J. Wisdom, S. Peale and F. Mignard, Icarus **58,** 137-152 (**May 1984**)
J. P. Paz and W. Zurek, PRL 75 351 (1995), W. Zurek Phys. Scr. **1998** 186
M. V. Berry, 'Chaos and the semiclassical limit of quantum mechanics' (2001)

1984 + 37 = 2021 !

Liouville correspondence

- Comparing evolution of *distributions* in phase space
- Classical: $\frac{\partial\rho(\mathbf{z},t)}{\partial t} = \{\rho(\mathbf{z},t), H(\mathbf{z})\}_{PB}$
- Quantum: $\frac{\partial W(\mathbf{z},t)}{\partial t} = \{W(\mathbf{z},t), H(\mathbf{z})\}_{MB} \simeq \{W(\mathbf{z},t), H(\mathbf{z})\}_{PB} + O(\hbar)$

Classical and quantum disagree when Wigner function becomes negative (typically, interference)

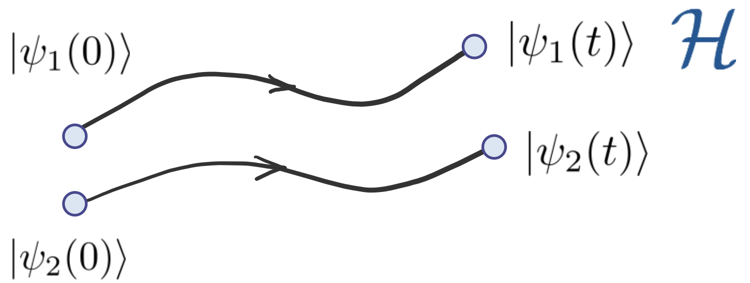
Correspondence is typically more accurate, but break times scale in the same way with $S_0/\hbar$

J. Emerson, PhD Thesis, Simon Fraser University (2001)

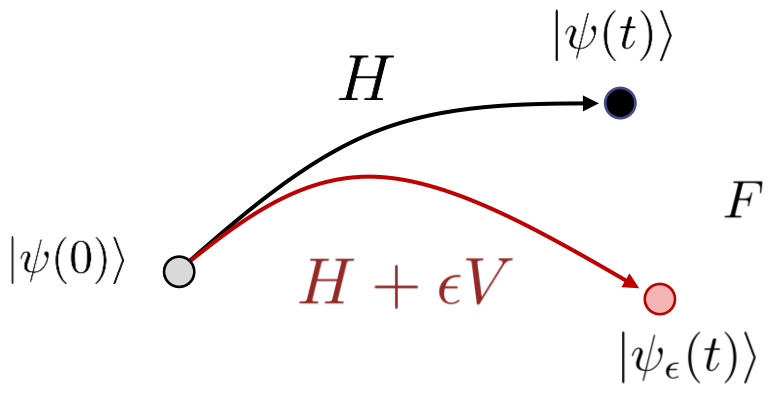
Chaotic dynamics makes classical states turn quantum very quickly!

Sensitivity to perturbations

- Evolution in Hilbert space is linear → trajectories can't separate 'exponentially'
- Unitarity implies $d_{12} = |\langle\psi_1(t)|\psi_2(t)\rangle| = |\langle\psi_1(0)|\psi_2(0)\rangle|$



Instead, look at small **deviations in the Hamiltonian:** A. Peres, *Stability of quantum motion,* Phys. Rev. A **30**, 1610 (1984)



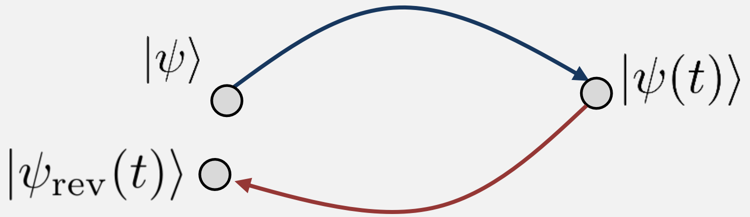
Fidelity

$$F(t) = |\langle\psi(t)|\psi_\epsilon(t)\rangle|^2$$

Loschmidt echo $F(t) = |\langle\psi_{\text{rev}}(t)|\psi\rangle|^2$

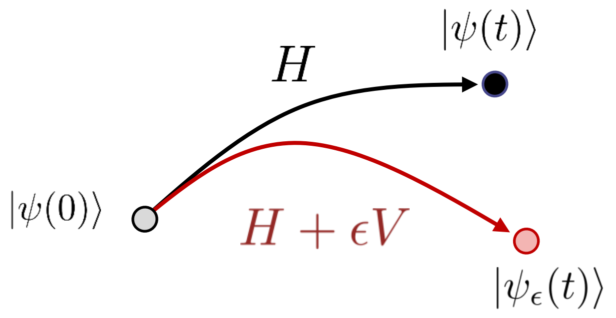
$$|\psi_{\text{rev}}(t)\rangle = e^{i(H+\epsilon V)t}e^{-iHt}|\psi\rangle$$

backwards forward



Simple model for fidelity decay

Based on A. Peres, *Stability of quantum motion,* Phys. Rev. A **30**, 1610 (1984)



Eigenstuff of H

$$|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle = \sum_k \langle\phi_k|\psi_0\rangle e^{-iE_k t}|\phi_k\rangle$$

$$|\psi_\epsilon(t)\rangle = e^{-i(H+\epsilon V)t}|\psi_0\rangle = \sum_k \langle\tilde{\phi}_k|\psi_0\rangle e^{-i\tilde{E}_k t}|\tilde{\phi}_k\rangle$$

Eigenstuff of $H + \epsilon\, V$

$$\boxed{\begin{aligned} |\tilde{\phi}_k\rangle &\simeq |\phi_k\rangle + \epsilon|\phi_k^{(1)}\rangle \\ \tilde{E}_k &\simeq E_k + \epsilon V_{kk} \end{aligned}}$$

$$F(t) = |\langle\psi(t)|\psi_\epsilon(t)\rangle|^2 = \left|\sum_k |\langle\phi_k|\psi_0\rangle|^2 e^{-i\epsilon V_{kk} t}\right|^2 + \mathcal{O}(\epsilon)$$

$$= \underbrace{\sum_k |\langle\phi_k|\psi_0\rangle|^4}_{} + \sum_{k\neq l} |\langle\phi_k|\psi_0\rangle|^2 |\langle\phi_l|\psi_0\rangle|^2 \boxed{e^{-i\epsilon(V_{kk}-V_{ll})t}}$$

Assume V is a Gaussian random matrix

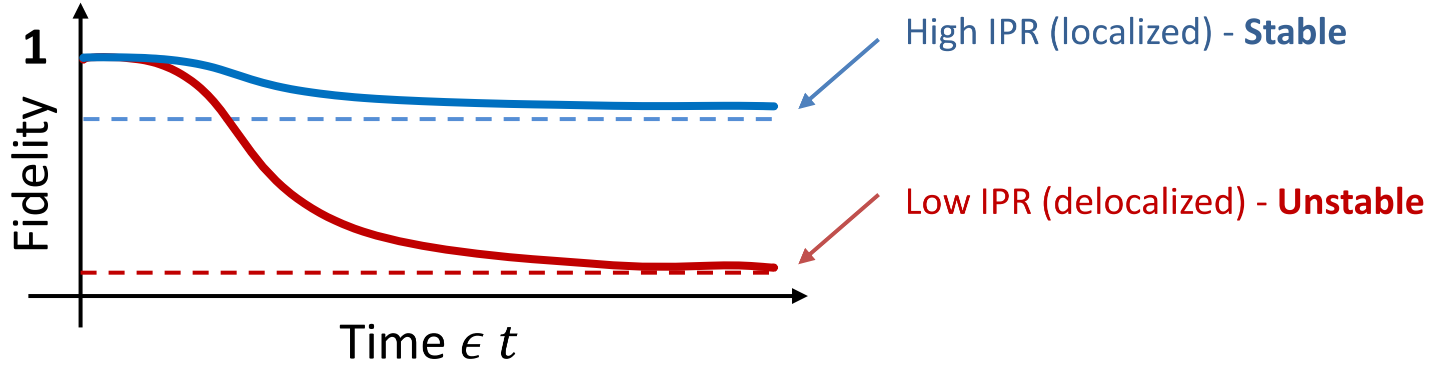
$$\langle\exp(-i\, V_{ii}\epsilon t)\rangle_V = \exp(-\frac{\epsilon^2 t^2}{2})$$

So, $\boxed{F(t) \simeq \eta_{\mathrm{IPR}} + e^{-\epsilon^2 t^2}(1-\eta_{\mathrm{IPR}})}$

Inverse participation ratio (IPR)

$\eta_{IPR} = \sum_k |\langle\phi_k|\psi\rangle|^4$: measures how localized a state is in a given basis

Notice: here $|\psi\rangle$ is the initial state, and $\{|\phi_k\rangle\}$ are the eigenstates of H



Fidelity decay and Loschmidt echo

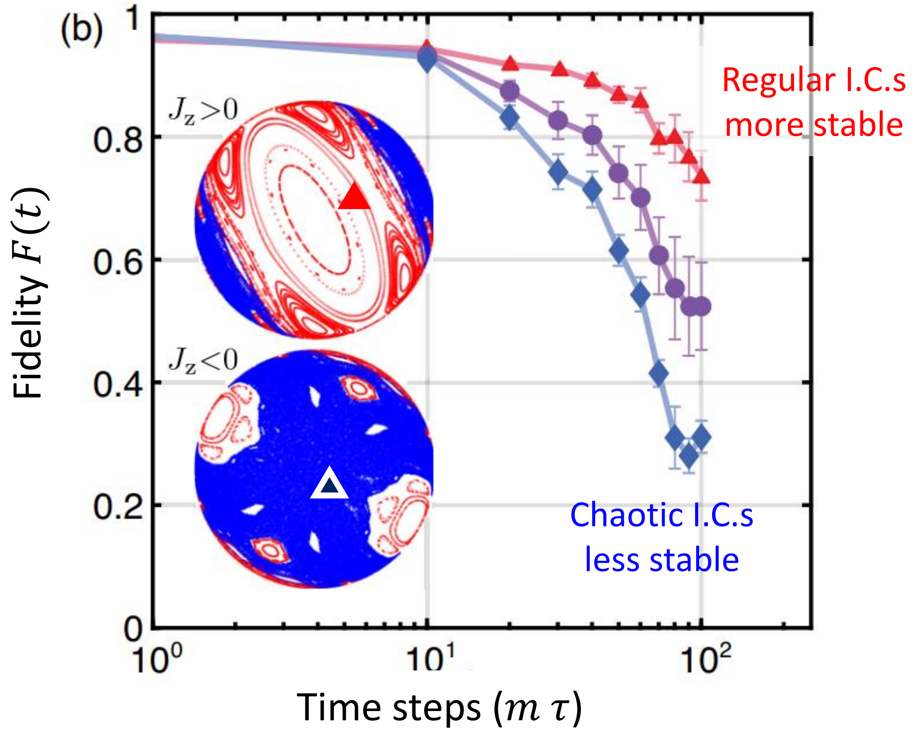
Recall the kicked top: $H = \frac{\alpha}{\tau} J_z + \frac{k}{2J} f(t) J_z^2$ $\quad f(t) = \sum_{m=-\infty}^{m=+\infty} \delta(t - m\tau)$

▲ Initial condition in regular part – high IPR (localized in energy)

▲ Initial condition in chaotic part – small IPR (delocalized in energy)

Intermediate values of k giving a 'mixed' phase space (with both regular and chaotic structures)

Experimental results using cold atoms – Poul Jessen's lab U. Arizona



From N. Lysne et al, Phys. Rev. Lett. **124** 230501 (2020)

Decay of the Loschmidt echo

- Gaussian decay is typical of the perturbative regime, breaks down in the semiclassical limit of chaotic systems
- There, the decay is typically **exponential.** The rate is perturbation-dependent first (intermediate ϵ), and then perturbation-independent, and given by the **largest Lyapunov exponent.**

R. Jalabert and H. Pastawski, Phys. Rev. Lett. **86**, 2490 (2001)
A. Goussev et al, Scholarpedia **7**, 11687 (2012)

Relation to OTOCs

- OTOCs - Newly rediscovered metrics for quantum chaos are also known to show intrinsic Lyapunov decay independently of the presence of a perturbation

I. García-Mata et al, Phys. Rev. Lett. **121**, 210601 (2018)

Session 6 (July somethingth):
OTOCs and scrambling!

Summary

- The behavior of the level spacing statistics is usually considered as the defining feature of quantum chaos. In the semiclassical regime, the BT and BGS conjectures provide formal links between quantum and classical integrability and chaos.
- Properties of eigenstates are also an important tool to diagnose quantum chaos. When expanded on a 'physical' basis, chaos can be interpreted as the average delocalization of energy eigenstates.
- Signatures of chaos in the dynamics of quantum systems can be seen through i) the fast breakdown of the quantum-to-classical correspondence (Ehrenfest time) and ii) the sensitivity of the evolution of a quantum state to small deviations in the Hamiltonian (fidelity decay)

Next week (June 16th): Random Matrix Theory (Changhao Yi) –
Main reference: Haake's book, chapter 4

References

- S. Wimberger, *Nonlinear dynamics and quantum chaos: an Introduction* (Chap. 4)
- F. Haake, *Quantum signatures of chaos* (Chap. 2 and 3)
- J. Emerson, PhD Thesis: *Quantum chaos and quantum-classical correspondence*
- A. Gubin and L. Santos, *Quantum chaos: An introduction via chains of interacting spins ½.* Am. J. Phys. **80,** 246 (2012)

Further reading

- D. Poulin, *A rough guide to quantum chaos* - *https://epiq.physique.usherbrooke.ca/pdf/Pou02a.pdf*
- M. V. Berry, https://michaelberryphysics.files.wordpress.com/2013/07/berry337.pdf
- J. P. Paz and W. Zurek, *Quantum chaos: a decoherent definition,* Physica D **83** 300-308 (1995)

Extra stuff

WKB approximation

↳ WKB approximates well eigenenergies of bound states

↳ $\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]E = E\psi(x)$

↳ ansatz $\psi(x)=e^{\frac{i}{\hbar}g(x)}$ $\in\mathbb{C}$

$$\Rightarrow \underbrace{-\frac{i\hbar}{2m}\frac{d^2g(x)}{dx^2}}_{O(\hbar)\ \text{"correction"}} + \underbrace{\frac{1}{2m}\left(\frac{dg(x)}{dx}\right)^2 + V(x) = E}_{\text{H-J equation}}$$

So, $\frac{dg}{dx} = \pm\sqrt{p^2(x)+i\hbar\frac{d^2g}{dx^2}}$ $\quad p(x)=\sqrt{2m(E-V(x))}$

obs: $\frac{dg}{dx}=\pm p(x)+O(\hbar) \Rightarrow \frac{d^2g}{dx^2}=\pm\frac{dp}{dx}+O(\hbar)$

$$= \mp\frac{m}{\sqrt{p(x)}}\frac{dV}{dx}+O(\hbar)$$

$$\Rightarrow \frac{dg}{dx}=\pm\sqrt{p^2(x)\mp i\hbar\frac{dp}{dx}}$$

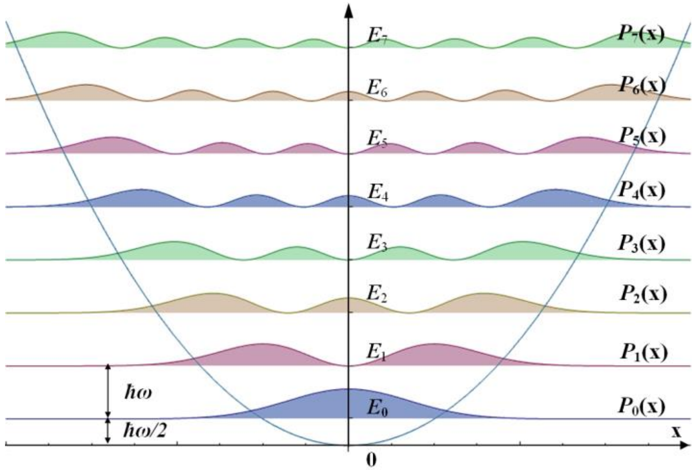
$$\simeq \pm p(x) + i\hbar\frac{1}{2p(x)}\frac{dp}{dx}+O(\hbar^2)$$

$$\Rightarrow g(x)=\pm\int_{x_0}^{x} p(x')\,dx' + \frac{i\hbar}{2}\log(p(x))$$

$$\Rightarrow \boxed{\psi(x)=\frac{A}{\sqrt{p(x)}}e^{\pm\frac{i}{\hbar}\int_{x_0}^{x} p(x')dx'} \quad \text{WKB wave-function}}$$

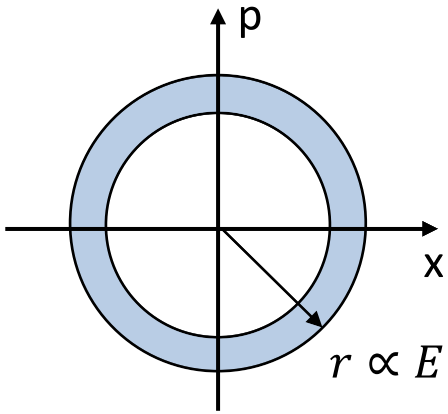
Highest weight near turning points

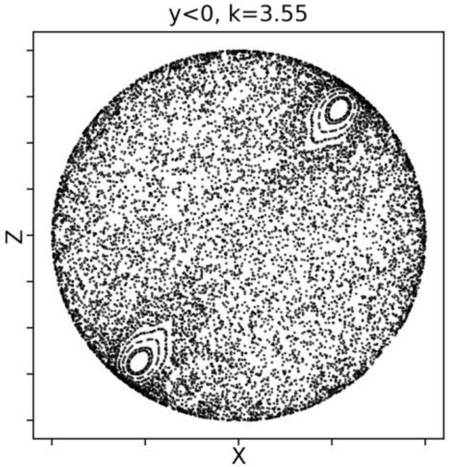
$|\psi(x)|^2\,\Delta x \propto$ time spent in a given interval $[x, x+\Delta x]$



Harmonic oscillator

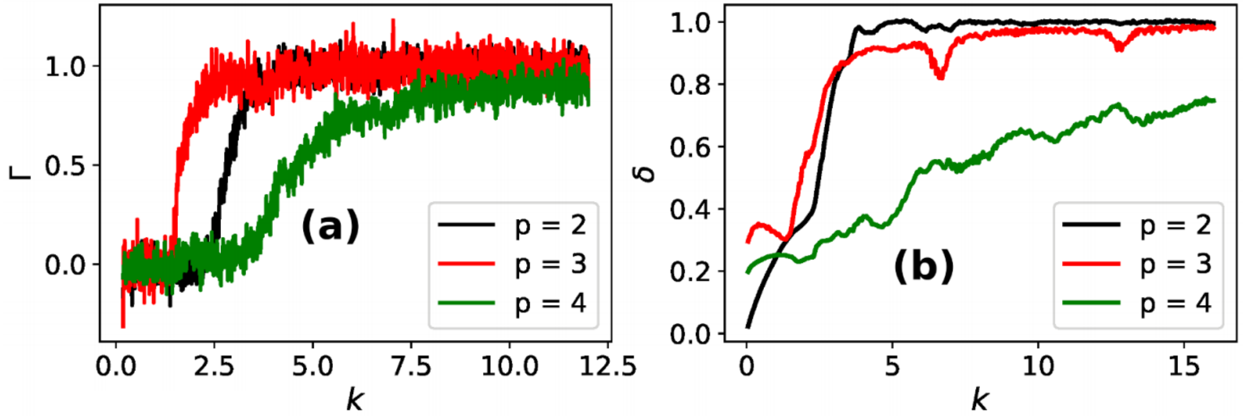
(taken from wiki https://en.wikipedia.org/wiki/Quantum_harmonic_oscillator)



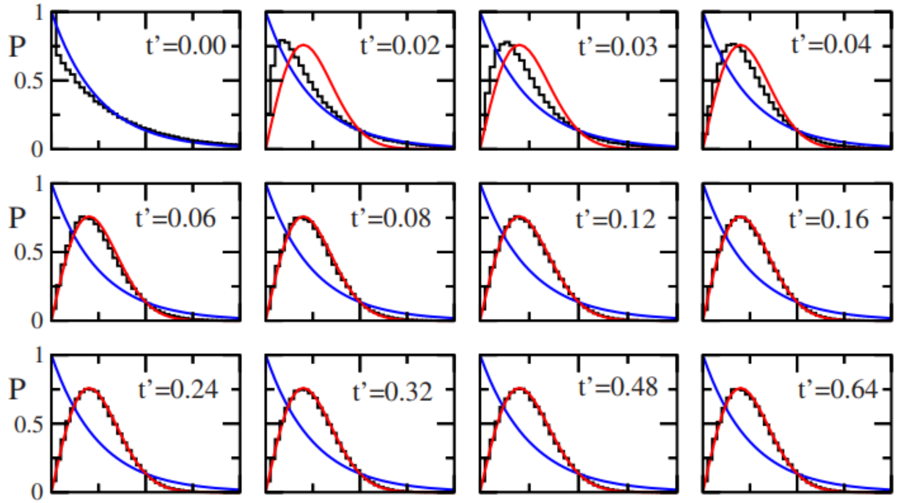


Chaotic trajectories have fixed energy, but are extremely delocalized in phase space!

Ergodicity – they spend roughly the same amount of time everywhere



$$H_B = \sum_{i=1}^{L} \Big[-t(b_i^\dagger b_{i+1} + \text{H.c.}) + V\left(n_i^b - \frac{1}{2}\right)\left(n_{i+1}^b - \frac{1}{2}\right) - t'(b_i^\dagger b_{i+2} + \text{H.c.}) + V'\left(n_i^b - \frac{1}{2}\right)\left(n_{i+2}^b - \frac{1}{2}\right) \Big],$$



From L. F. Santos and M. Rigol, Phys. Rev. E 81 036206 (2010)

Hilbert space for the classical Liouville equation.— Consider the time-independent classical Hamiltonian $H(\mathbf{z})$, where $\mathbf{z} = (x_1, \ldots, x_N, p_1, \ldots, p_N)$ specifies a point in N-dimensional phase space and $H(\mathbf{z})$ belongs to an integrable or nonintegrable system. The evolution of the phase space distribution $\rho(\mathbf{z}, t)$ obeys the classical Liouville equation:

$$i\frac{\partial\rho(\mathbf{z}, t)}{\partial t} = \hat{L}\rho(\mathbf{x}, t) \equiv i\{H(\mathbf{z}), \rho(\mathbf{z}, t)\}, \tag{5}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket, $\hat{L}$ is called the Liouvillian, and $\rho(\mathbf{z}, t)$ is normalized as $\int d\mathbf{z}\rho(\mathbf{z}, t) = 1$. The Liouvillian is a Hermitian operator with respect to the given inner product $\langle\rho_1|\rho_2\rangle \equiv \int d\mathbf{z}\rho_1^*(\mathbf{z})\rho_2(\mathbf{z})$. Then, using the eigenstate $|n\rangle$ of $\hat{L}$, we can expand $\rho(\mathbf{z}, t)$ as

$$|\rho(t)\rangle = \sum_n c_n e^{-i\lambda_n t}|n\rangle, \tag{6}$$

where c_n is a time-independent constant and $\langle\rho(t)|\rho(t)\rangle \neq 1$. We note that $\langle\rho|\hat{L}|\rho\rangle = 0$ and, if λ_n is an eigenvalue of $\hat{L}$, then $-\lambda_n$ is also an eigenvalue of $\hat{L}$ (see Supplemental Material [34] for the proof).

In the following, we obtain the CSL for the classical Liouville equation by using the Hilbert space for the classical Liouville equation.