Quantum Integrability in two parts

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1) Part one: the notion of Quantum Integrability? [Manuel]

2) Part two: On exact solutions of quantum integrable systems [Austin]

Part one: The notion of Quantum Integrability

Classical integrability

Consider a classical Hamiltonian system with N degrees of freedom and $2N$-dimensional phase space Γ, described by a Hamiltonian $H(\mathbf{q},\mathbf{p})$, we say the system is integrable if:

- There exist N single valued constants of motion $I_\alpha, \quad \alpha = 1, .., N$, defined smoothly over all phase space, thus

$$\{H, I_\alpha\} = 0, \;\; \alpha = 1, .., N \rightarrow \;\; \frac{dI_\alpha}{dt} = \{I_\alpha, H\} = 0.$$

- The N constants of motion are functionally independent

- All N constants of motion are in Involution, *i.e*

$$\{I_\alpha, I_\beta\} = 0, \;\; \text{with} \;\; \alpha \neq \beta$$

Classical integrability

Recall [**From Pablo's first lecture**], that, given the above conditions, there always is a canonical transformation such that

$$H(\mathbf{q}, \mathbf{p}) \to H(I_1, I_2, ..., I_N),$$

i.e, Hamiltonian is cyclic in the conjugated "momenta", and the equations of motion can be readily integrated

$$\frac{dI_\alpha}{dt} = \frac{\partial H}{\partial \Theta_\alpha} = 0, \quad \frac{d\Theta_\alpha}{dt} = -\frac{\partial H}{\partial I_\alpha} = \mathrm{Cnst}$$

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Why is this notion important/useful/interesting?

- Gives a clear physical picture to the term **integrable system:** Phase space is foliated by “hypersurfaces” of constant action which are difeomorphic to high-dimensional torus
- It divides the space of physical models in categories **which make sense!** The motion a model in each category undergoes is markedly different, **regular** (few well defines frequencies) vs **irregular** (dense frequency spectrum, mixing and the ergodic hierarchy)
- It allow us to make general statements about the long time evolution of observables, thermalization, and stability against small perturbations (KAM theorem), without the need of a case-by-case analysis

Quantum Integrability (QI)

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Ideally, we would like a notion of QI which allows us to make general statements regarding nonequilibrium dynamics, thermalization, and stability against small perturbations, among others!

An (incomplete), set of requirements is:

1) It should be unambiguous

2) It should partition the set of all possible quantum models into distinct classes

3) Different class of models should display distinguishable physical behavior

Stefan Weigert, *"The problem of quantum integrability",* Physica D, **56** 107-119 (1991)

Jean-Sébastien Caux and Jorn Mossel, *"Remarks on the notion of quantum integrability",* J. Stat. Mech. P02023 (2011)

QI, first attempt: Promoting the classical notion

We promote Poisson brackets to commutators $\{\ ,\ \} \to \frac{i}{\hbar}[\ ,\ ]$

Evolution of an observable

$$\frac{d}{dt}\hat{O} = -\frac{i}{\hbar}[\hat{O}, \hat{H}]$$

- We can then promote the classical notion to a quantum one!

A quantum system is integrable if it has maximal set of conserved charges*

$\{\mathcal{Q}_\alpha\}, \;\; \alpha = 1, ..., \dim(\mathcal{H})$. Satisfying: $[\mathcal{Q}_\alpha, \hat{H}] = [\mathcal{Q}_\alpha, \mathcal{Q}_\beta] = 0$

*short comings of this definition are discussed in: Stefan Weigert, *“The problem of quantum integrability”*, Physica D, **56** 107-119 (1991)

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- However, any Hermitian operator can be brought in to the form

$$\hat{H} = \sum_i E_i \hat{P}_{E_i}, \ \ \text{where}\ \ \hat{P}_{E_i} = |E_i\rangle\langle E_i|$$

Hence the choice $\mathcal{Q}_\alpha = \hat{P}_{E_\alpha}$ leads to all Hamiltonians being quantum integrable!

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Although this notion is strict in the classical case, it results insufficient in the quantum case!

QI, second attempt: trying to salvage attempt # 1

- Promoting the classical notion lead us to find that every quantum system has a maximal set of conserved charges.
- However, the task of finding them might not be easily executed. One can then try to go on and define

A quantum system is integrable if the associated eigenvalue problem that it defines can be exactly solved.
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- This notion is appealing, however it treats all exactly solvable models in the same footing regardless of the **method used to solve them** and the possible physical phenomena emerging on these models. For instance

A tight-binding Hamiltonian

$$\hat{H} = -J \sum_l (|l+1\rangle\langle l| + \mathrm{h.c}) + \epsilon \sum_l |l\rangle\langle l|$$

Solvable in Fourier space

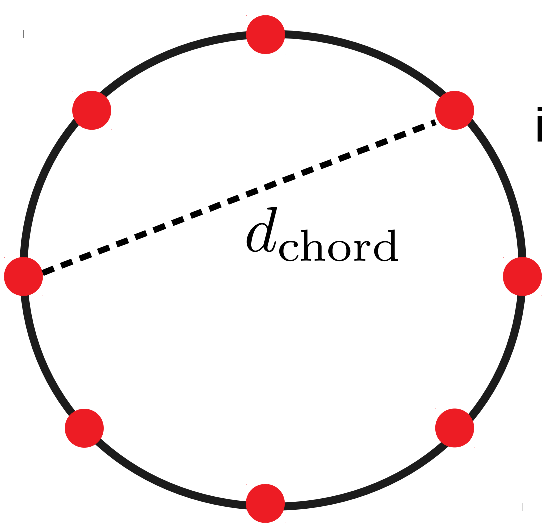
$$|k\rangle = \sum_l e^{ikl} |l\rangle$$

Would be as “integrable” as, for example, the *XXX*-spin chain!
[Solvable via the Algebraic Bethe Ansatz]

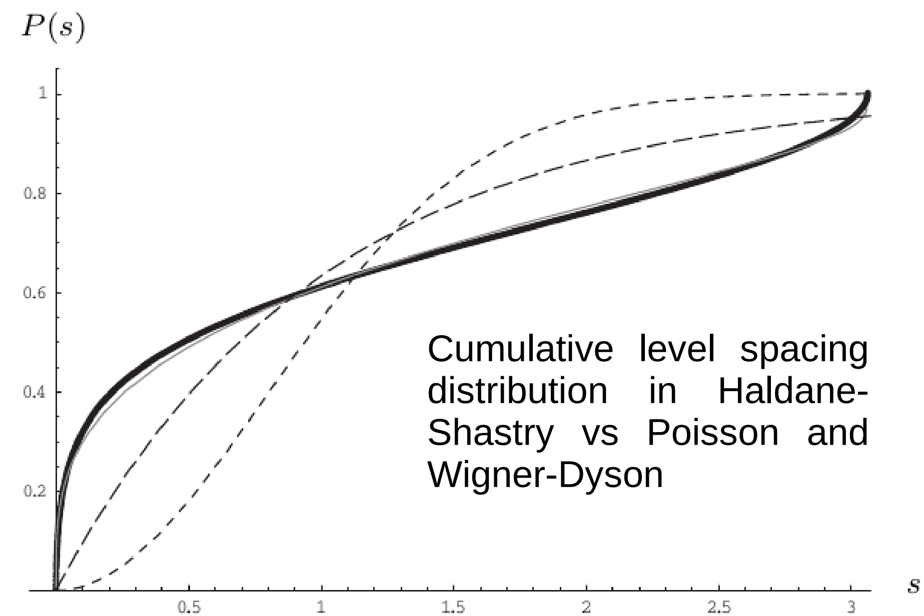
QI, some other possibilities

1) A quantum system is integrable if its spectral statistics follows that of a Poisson distribution

- For systems with a well defined classical limit, this notion ascribes to the Berry-Tabor conjecture.
- There, also, is large evidence of its relevance in many-body systems, however there are also counter examples, for instance spin chains of the Haldane-Shastry type*



interaction $\propto \dfrac{1}{d^2_{\mathrm{chord}}}$



Cumulative level spacing distribution in Haldane-Shastry vs Poisson and Wigner-Dyson

*see: Federico Finkel and Artemio González-Lopéz, *“Global properties of the spectrum of the Haldane-Shastry type”*, Phys. Rev. B., **72** 174411 (2005)

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2) A quantum system is integrable if its spectrum presents level crossings

- The problem with this notion is that it only allow us to make meaningful staments about parametrized families of models. It is useless for Hamiltonians at fixed parameter values.

Some others are discussed in: Jean-Sébastien Caux and Jorn Mossel, *“Remarks on the notion of quantum integrability”,* J. Stat. Mech. P02023 (2011)

Where all of this leave us?

- For a notion of quantum integrability, existence of conserved charges, even in the form of a maximal set, is insufficient. **The structure of conserved charges matters!**
  – The structure of operators depends on the basis we use to represent them. In short, a notion of quantum integrability will only have validity in a specified basis*.

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- If we want to enforce locality of the conserved charges, then the maximality of the set needs to be dropped. In fact, any maximal set of conserved charges will, unavoidable, contain nonlocal charges.
  - – It is now accepted that cardinality of the set

$$\mathcal{C}_\alpha = |\{\mathcal{Q}_\alpha\}| < \dim(\mathcal{H}), \text{ and } \mathcal{C}_\alpha \to \infty \text{ when } \dim(\mathcal{H}) \to \infty$$

  that is, becomes unbounded in the thermodynamic limit.
  In other words, a nonmaximal set of local/quasi-local conserved charges is still enough, as long as it becomes extensive in the thermodynamic limit**

*see: Jean-Sébastien Caux and Jorn Mossel, *"Remarks on the notion of quantum integrability",* J. Stat. Mech. P02023 (2011)

**see chapter one of: Pieter Claeys, *"Richardson-Gaudin models and broken integrability",* PhD dissertation Ghent University (2018)

Some final comments

- Analyzing the structure of the conserved charges enables us to split models into distinct classes. Caux and Mossel*, proposed a QI notion that achieves this via the *density character*: simplest function bounding the number of nonzero entries of the matrix representation of the charges
  – This allows them to talk about models which are, *constant integrable, linear integrable, polynomial integrable, etc.*
  – Any model in the exponential class or above is then referred to as nonintegrable

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Some final comments

Deviations from **ergodicity** are, generally, attributed to quantum integrability*.

Consider an operator $\hat{A}$ and the canonical average $\langle \Delta \hat{A}(0) \Delta \hat{A}(t) \rangle$ with $\Delta \hat{A} = \hat{A} - \langle \hat{A} \rangle$, the operator is called ergodic if

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \langle \Delta \hat{A}(0) \Delta \hat{A}(t) \rangle = 0.$$

The physical origin of deviations from the ergodic average and their connection with quantum integrability can be interpreted using Mazur's inequality

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \langle \Delta \hat{A}(0) \Delta \hat{A}(t) \rangle \geq \sum_{\alpha=1}^{N_{\tilde{\mathcal{Q}}}} \frac{|\langle (\Delta \hat{A} \tilde{\mathcal{Q}}_\alpha)|^2}{\langle \tilde{\mathcal{Q}}_\alpha^2 \rangle}$$

As any nonnegligible "overlap" between the observable and one of the conserved charges will lead to a deviation from the ergodic average.

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summary

- The classical notion of integrability is insufficient in the context of quantum systems
- Quantum integrability has validity only in a specified basis
- The set of conserved charges does not need to be maximal. A set composed of local and quasi-local charges is enough provided it is extensive in the infinite size limit
- Quantum integrability can be viewed as deviations from ergodicity. However, many subtleties arise, type of averaging used, structure of observables, etc. We will look more into this during next week's lecture [by Sam and Mason]

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Thanks!