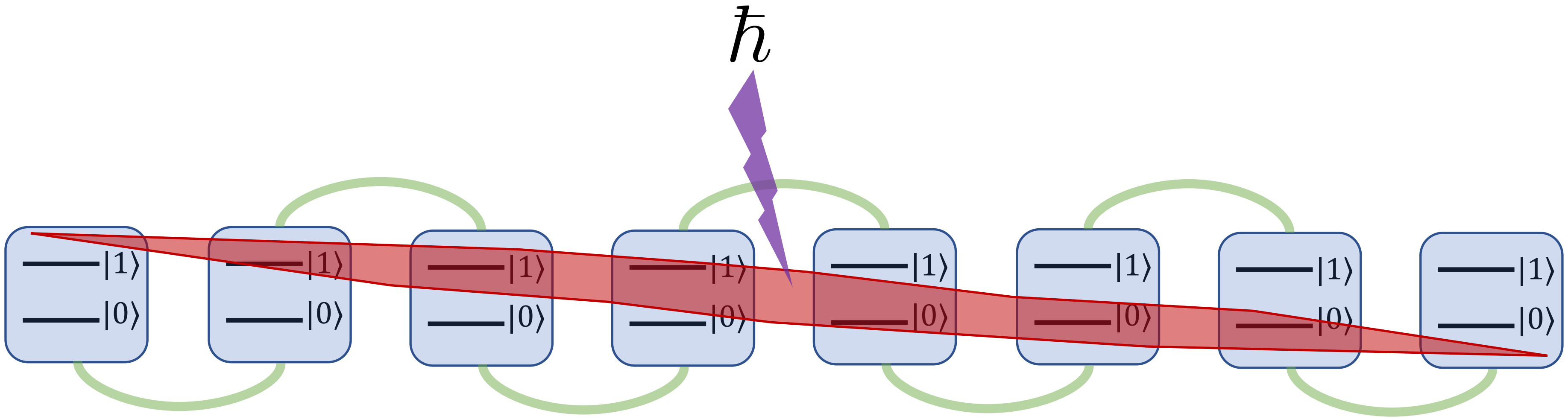
Quantum Many-Body Scars



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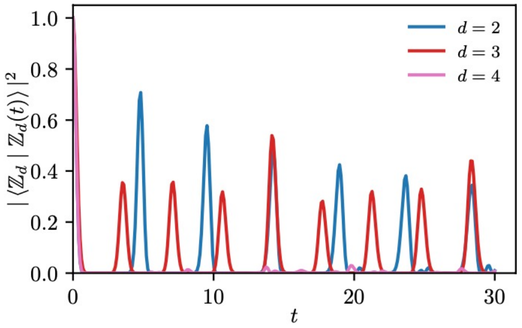
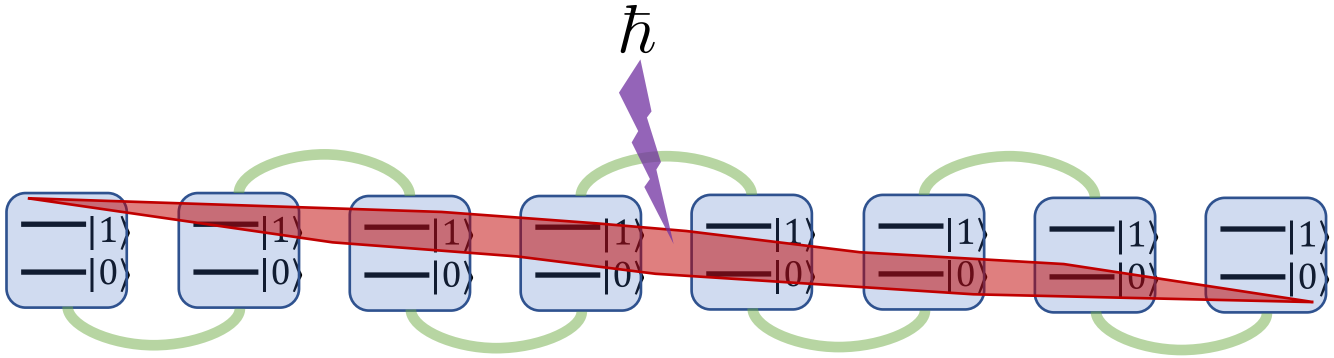
Qchaos Summer Study

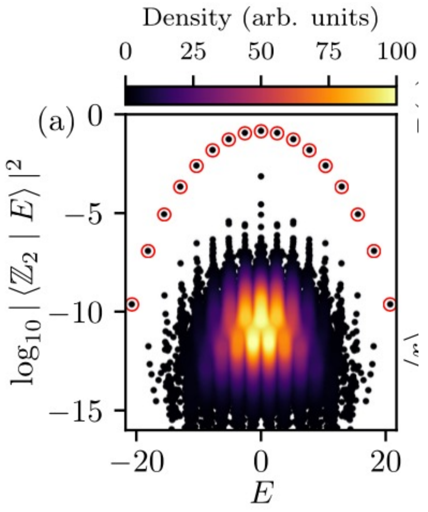
127.0.0.1

Quantum many-body scars

- Weak breakdown of thermalization in quantum chaotic Hamiltonians.
- Named after the phenomenon of “quantum scars” seen in eigenfunctions of chaotic Hamiltonians
- First experimentally observed in a 1D array of neutral atoms with Rydberg excitation.
- Simple but rich model – PXP Hamiltonian for 1D array of neutral atoms.







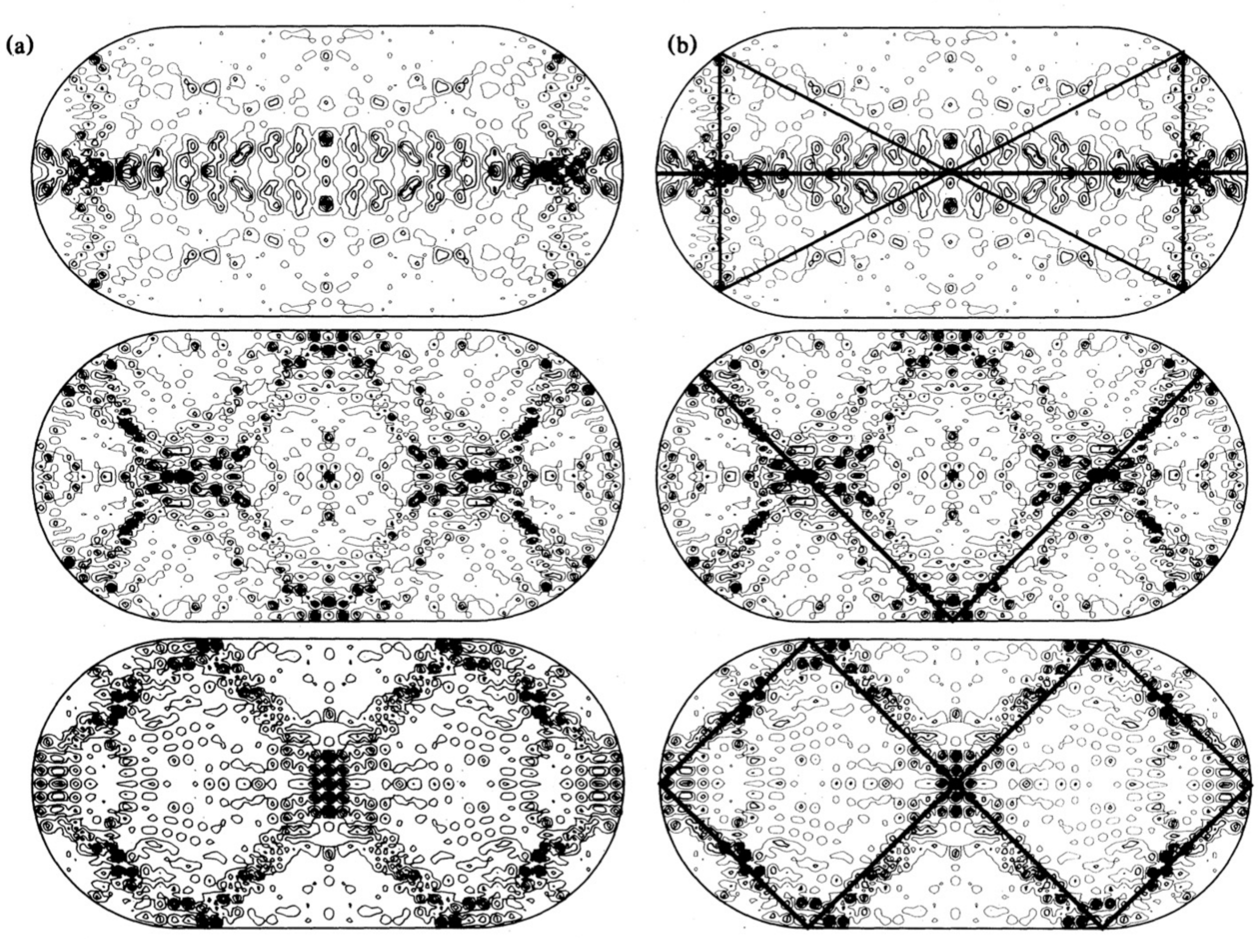
Quantum Scars – Hamiltonian Eigenstates

Unstable periodic orbits induce scars of larger than expected density in some eigenstates of the Hamiltonian.

Example: Bunimovich stadium

- Left: probability density for “scarred” eigenstates of the Hamiltonian.
- Right: Unstable periodic orbits which scar these eigenstates.

We can construct quasi-eigenstates using Gaussian wave packets centered on points along unstable periodic orbit.



Phys. Rev. Lett. 53, 1595 (1984)

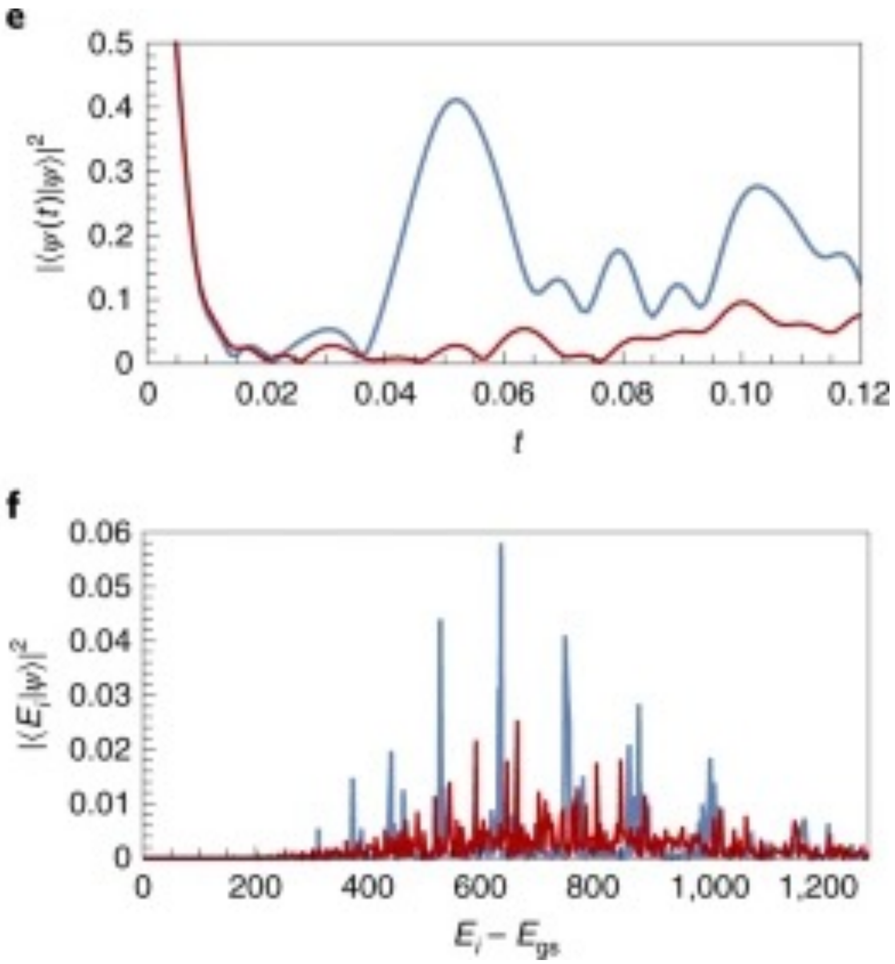
Quantum Scars – Dynamics

Time evolution: $\mathrm{e}^{-\mathrm{i}Ht/\hbar}$

Example: Bunimovich stadium with Gaussian wave packet as the initial state.

- (Cyan) Along an unstable periodic orbit.
- (Dark red) Off an unstable periodic orbit by angle $\pi/4$.
- Revival probability in the time domain.
    - Gaussian wave packet along an unstable period orbit has revivals.
    - Gaussian wave packet not along an unstable period orbit has negligible revivals.
- Overlap of initial state with energy eigenstates.
    - Gaussian wave packet along an unstable period orbit has periodic sequence of peak in overlaps with energy eigenstates
    - Gaussian wave packet not along an unstable period orbit has a continuum of frequencies in overlaps with energy eigenstates.

Gaussian wave packet along unstable period orbit => quasi-eigenstate.



arxiv:2011.09486

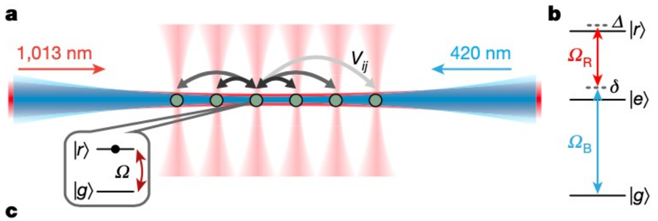
Quantum Many-Body Scars – Experimental discovery

ARTICLE

doi:10.1038/nature24622

Probing many-body dynamics on a 51-atom quantum simulator

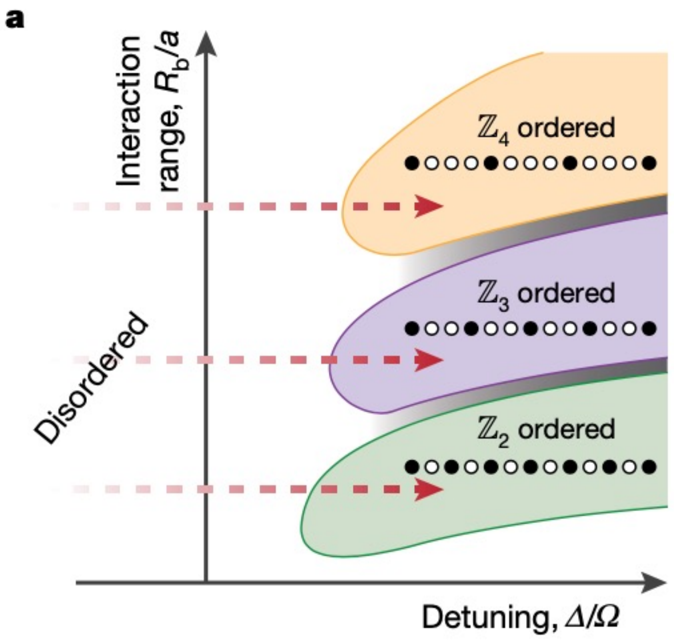
Hannes Bernien[1], Sylvain Schwartz[1,2], Alexander Keesling[1], Harry Levine[1], Ahmed Omran[1], Hannes Pichler[1,3], Soonwon Choi[1], Alexander S. Zibrov[1], Manuel Endres[4], Markus Greiner[1], Vladan Vuletić[2] & Mikhail D. Lukin[1]

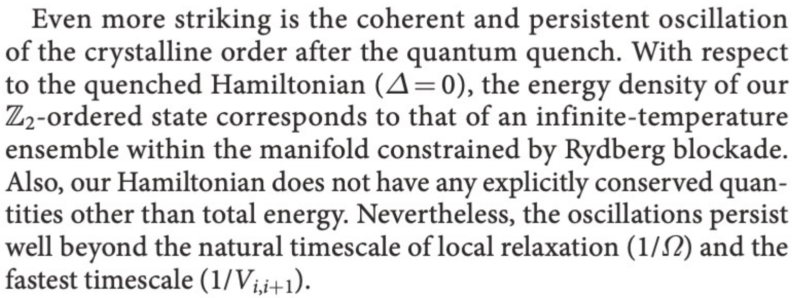


$$\frac{1}{\hbar}H = \sum_j \left(\frac{\Omega}{2}X_j - \Delta Q_j\right) + \sum_{j_1<j_2} V_{j_1,j_2} Q_{j_1} Q_{j_2}$$

$$Q_j = \frac{\mathbb{1} + Z_j}{2} = |r\rangle\langle r| = |\bullet\rangle\langle\bullet|$$

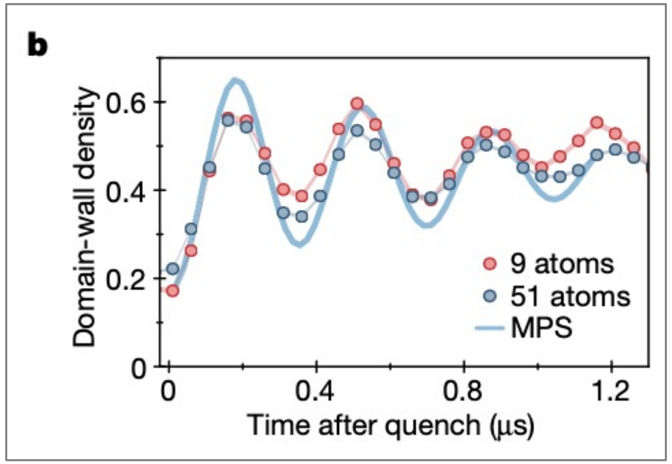
$$V_{j_1,j_2} \propto |j_1 - j_2|^{-6}$$





Even more striking is the coherent and persistent oscillation of the crystalline order after the quantum quench. With respect to the quenched Hamiltonian ($\Delta = 0$), the energy density of our $\mathbb{Z}_2$-ordered state corresponds to that of an infinite-temperature ensemble within the manifold constrained by Rydberg blockade. Also, our Hamiltonian does not have any explicitly conserved quantities other than total energy. Nevertheless, the oscillations persist well beyond the natural timescale of local relaxation ($1/\Omega$) and the fastest timescale ($1/V_{i,i+1}$).

Quench from $\mathbb{Z}_2$ to $\Delta = 0$



Surprising revivals when initial state is $\mathbb{Z}_2$.

$$|\mathbb{Z}_2\rangle = |\cdots \bullet \circ \bullet \circ \bullet \circ \cdots\rangle$$

arxiv:1707.04344

PXP Hamiltonian for Rydberg atom arrays

$$\frac{1}{\hbar}H = \sum_j \left(\frac{\Omega}{2}X_j - \Delta Q_j\right) + \sum_{j_1<j_2} V_{j_1,j_2}Q_{j_1}Q_{j_2}$$

$$V_{j_1,j_2} \propto |j_1 - j_2|^{-6}$$

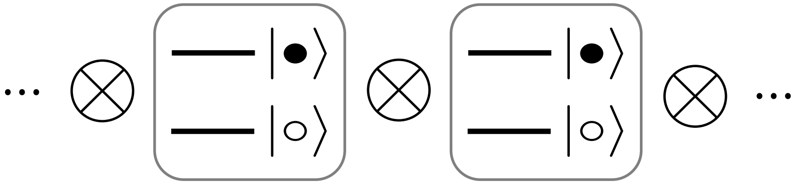
$$Q_j = \frac{\mathbb{1} + Z_j}{2} = |r\rangle\langle r| = |\bullet\rangle\langle\bullet|$$

$$P_j = \frac{\mathbb{1} - Z_j}{2} = |g\rangle\langle g| = |\circ\rangle\langle\circ|$$

$$H_{\mathrm{PXP}} = \sum_j P_{j-1}X_jP_{j+1}$$

$$X = |\bullet\rangle\langle\circ| + |\circ\rangle\langle\bullet|$$

Tensor product
Hilbert space



$$\hbar = 1$$
$$|V_{j,j+1}| \gg \Omega \gg |\Delta|$$

Nearest neighbor Rydberg blockade:
remove states with adjacent Rydberg
excitations.

$$|\cdots \xcancel{\bullet\bullet} \cdots\rangle$$

“Fibonacci chain”, “PXP Hamiltonian”: model for studying the statics, dynamics of the Rydberg atom system

arxiv:1711.03528, arxiv:1806.10933

PXP Hamiltonian – Hilbert Space Structure

Nearest neighbor two-site configurations allowed $\{\circ\circ, \circ\bullet, \bullet\circ\}$

Open boundary conditions (OBC)

- Configurations can end with ground, $\circ$ or Rydberg, $\bullet$.
- L-atom configurations ending with Rydberg $\cdots\bullet$ are obtained by appending $\circ\bullet$ to $(L-2)$-atom configurations.
- L-atom configurations ending with Rydberg $\cdots\circ$ are be obtained by appending $\circ$ to $(L-1)$-atom configurations.
- This gives the recursion relation for Hilbert Space dimension for L-atom configurations, d_L

$$d_L = d_{L-1} + d_{L-2}$$

- Initial condition for recurrence

$$d_0 = 1,\ d_1 = 2$$

- Hilbert space dimension for L-atom configuration with OBC with the $(L+2)$the Fibonacci number

$$d_L = F_{L+2}$$

Periodic Boundary Conditions (PBC)

- L-atom configurations for PBC involves taking L-atom configurations and removing all configurations that begin and end with the Rydberg state $\bullet\cdots$ and $\cdots\bullet$.
- This gives the recursion relation for Hilbert Space dimension

$$d_L^{\mathrm{PBC}} = d_L - d_{L-4}$$

- Hilbert space dimension for L-atom configuration with PBC is the sum of $(L-1)$th and $(L+1)$th Fibonacci numbers

$$d_L^{\mathrm{PBC}} = F_{L-1} + F_{L+1}$$

- Hilbert Space dimension is related to Fibonacci numbers. "Fibonacci chain".
- $d_L \sim \varphi^L = \left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^L$
- $d_L < 2^L$, but still exponential!

arxiv:1711.03528, arxiv:1806.10933

PXP Hamiltonian – Symmetries

$$H_{\mathrm{PXP}} = \sum_j P_{j-1} X_j P_{j+1}$$

$$X = |\bullet\rangle\langle\circ| + |\circ\rangle\langle\bullet|$$

$$P_j = \frac{\mathbb{1} - Z_j}{2} = |g\rangle\langle g| = |\circ\rangle\langle\circ|$$

- Discrete spatial inversion symmetry.
  - Inversion symmetric sector, labelled 0 or $+$.
  - Inversion anti-symmetric sector, labelled π or $-$.
- “Particle-hole” symmetry in the many body spectrum.
  - Operator anti-commuting with the Hamiltonian.
  - Eigenstate with energy $+E$ has a partner eigenstate with energy $-E$.
- Translational symmetry for periodic boundary conditions (PBC).
  - Block diagonal in quasi momentum, k.

$$j \leftrightarrow L - j$$

$$\mathcal{C} = \bigotimes_j Z_j$$

- We typically use the quasi momentum, $k = 0$, inversion-symmetric, $+$ sector to analyze the Hamiltonian with PBC

arxiv:1711.03528, arxiv:1806.10933

PXP Hamiltonian – Energy Level Spacing

$$H_{\mathrm{PXP}} = \sum_j P_{j-1} X_j P_{j+1}$$

$$X = |\bullet\rangle\langle\circ| + |\circ\rangle\langle\bullet| \qquad P_j = \frac{\mathbb{1} - Z_j}{2} = |g\rangle\langle g| = |\circ\rangle\langle\circ|$$

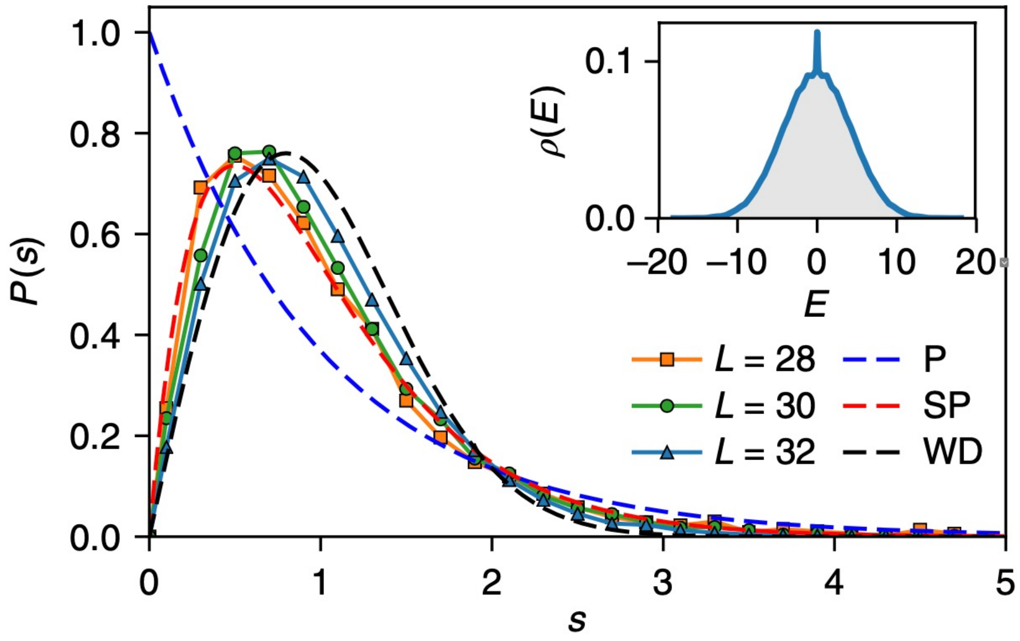
- Level spacing distribution, $P(s)$, approaches Wigner Dyson distribution for GOE as system size L is increased.
- Density of states, $\rho(\mathrm{E})$, has a Gaussian form, with a spike of density at $E = 0$ due to zero energy modes.

$$\mathscr{P}_{\text{Poisson}}(s) = \exp(-s)$$

$$\left.\begin{aligned}\mathscr{P}_{\text{Semi-Poisson}}(s) &= 4s\exp(-2s)\\ \mathscr{P}_{\text{WD-GOE}}(s) &= \frac{\pi}{2}s\exp\left(-\frac{\pi}{4}s^2\right)\end{aligned}\right\}\text{Have level repulsion}$$

- PXP Hamiltonian is **chaotic**!
- Eigenstates are not straightforward to find.
- We will find quasi-eigenstates.

Level spacing statistics in the zero quasi momentum, inversion symmetric sector



P: Poisson
SP: Semi Poisson
WD: Wigner Dyson for GOE

arxiv:1711.03528, arxiv:1806.10933

Creating quasi-eigenstates – simple example

$$H_{\text{paramagnet}} = \sum_j X_j = \sum_j \sigma_j^+ + \sum_j \sigma_j^-$$

Adds excitation to site j, summed over all sites

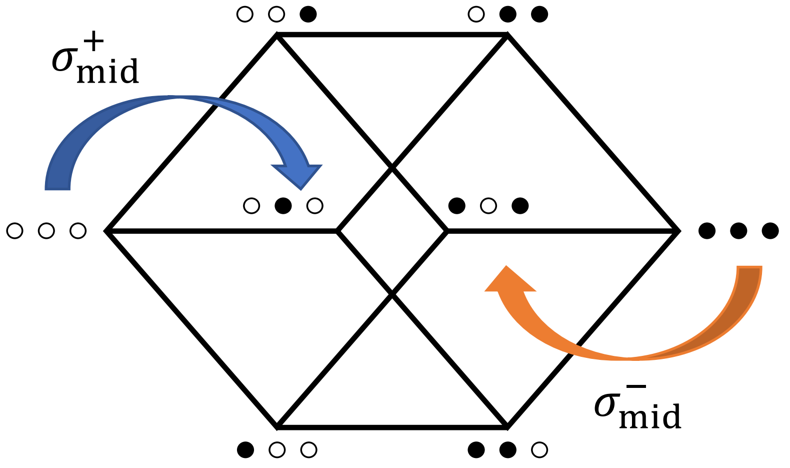
Removes excitation from site j, summed over all sites

$$H_{\text{paramagnet}} = \sum_j X_j = H_+ + H_-$$

Collective spin raising operator

Collective spin lowering operator

Z eigenbasis states in a hyper cube of 2^L dimensions. Here $L = 3$.



σ_j^+, σ_j^- move around the Z eigenbasis in a Gray code fashion

Z eigenstates are **quasi eigenstates** of this Hamiltonian**.**

arxiv:1711.03528, arxiv:1806.10933

Creating quasi eigenstates – PXP Hamiltonian

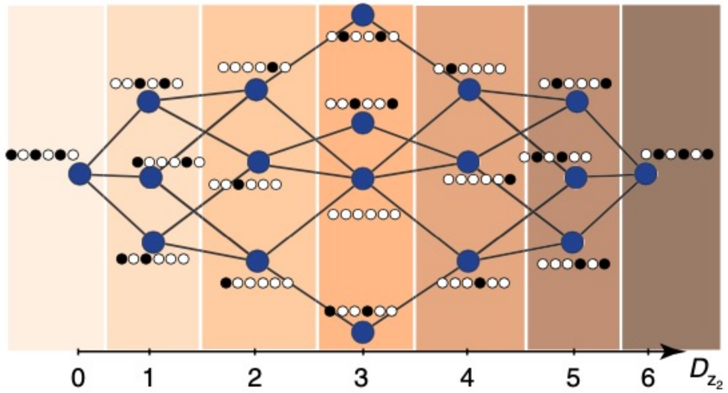
$$H_{\text{PXP}} = \sum_j P_{j-1} X_j P_{j+1} = H_+ + H_-$$

$$H_\pm = \sum_{j\in\text{even}} P_{j-1}\sigma_j^{\pm} P_{j+1} + \sum_{j\in\text{odd}} P_{j-1}\sigma_j^{\mp} P_{j+1}$$

Increases Hamming distance from the $\mathbb{Z}_2$ state

Decreases Hamming distance from the $\mathbb{Z}_2$ state

Low energy PXP model subspace for $L = 6$



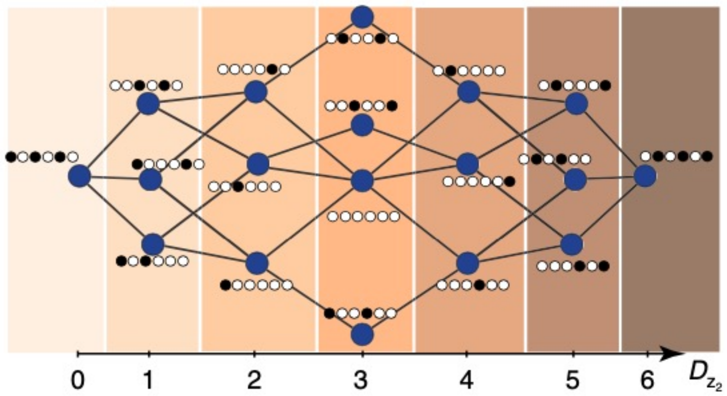
- These state are **quasi eigenstates** approximations for the PXP Hamiltonian.
- These are **not eigenstates** of the PXP Hamiltonian.

arxiv:1711.03528, arxiv:1806.10933

Creating quasi eigenstates – PXP Hamiltonian

Low energy PXP model subspace for $L = 6$

$$|\mathbb{Z}_2\rangle = |\bullet \circ \bullet \circ \bullet\circ\rangle$$



Hamming distance from $\mathbb{Z}_2$, $D_{\mathbb{Z}_2}$

$$|\mathbb{Z}_2'\rangle = |\circ \bullet \circ \bullet \circ\bullet\rangle$$

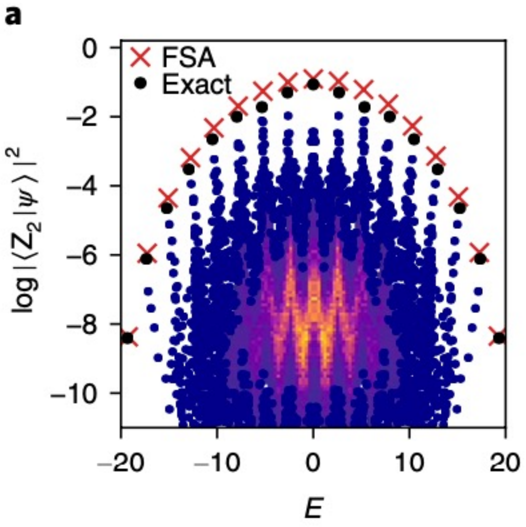
The Hamming distance from the $\mathbb{Z}_2$ state counts number of excitations, say $n \in \{0, L\}$

$$|n\rangle \propto \left(H^+\right)^n |\mathbb{Z}_2\rangle$$
$$|L-n\rangle \propto \left(H^-\right)^n |\mathbb{Z}_2'\rangle$$

- These state are **quasi eigenstates** approximations for the PXP Hamiltonian.
- These are **not eigenstates** of the PXP Hamiltonian.
- This is the **Forward Scattering Approximation** (FSA).

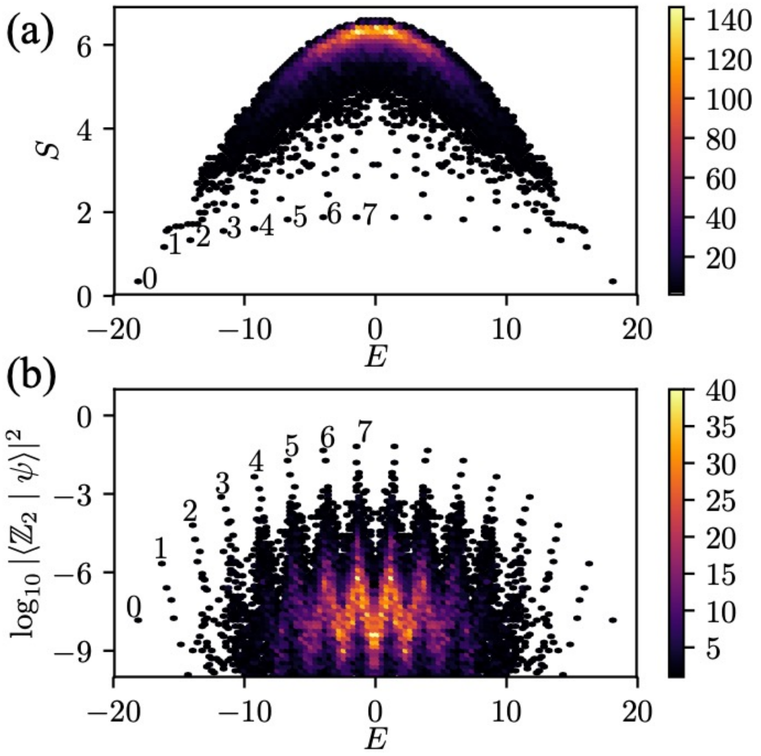
arxiv:1711.03528, arxiv:1806.10933

PXP Hamiltonian – Spectrum and special states



Density plot of overlap of energy eigenstates with $\mathbb{Z}_2$ as a function of energy.

- Band of special eigenstates separated from other eigenstates.
- Crosses denote overlaps with states calculated using Forward Scattering Approximation.
- Tower structure in overlaps



$$|\mathbb{Z}_2\rangle = |\cdots \bullet \circ \bullet \circ \bullet \circ \cdots\rangle$$

Density plot of half chain entanglement entropy of energy eigenstates as a function of energy.

- Band of special low entanglement eigenstates, for example 0, …, 7.

Density plot of overlap of energy eigenstates with $\mathbb{Z}_2$ as a function of energy.

- Band of special low entanglement states (0, …, 7) are the states with high overlap!

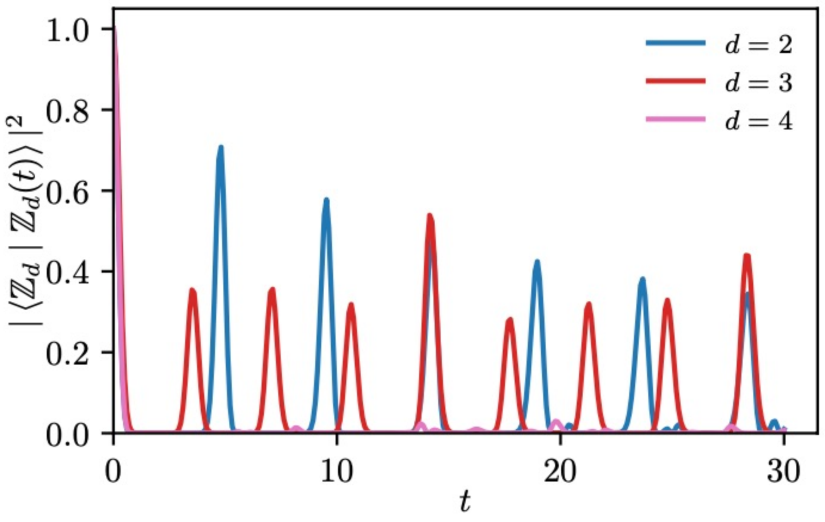
Special eigenstates (0, …, 7) span the entire energy range from lowest energy to middle. This model has energies in pairs $\pm E$.

arxiv:1711.03528, arxiv:1806.10933

Dynamics under PXP Hamiltonian – Revivals

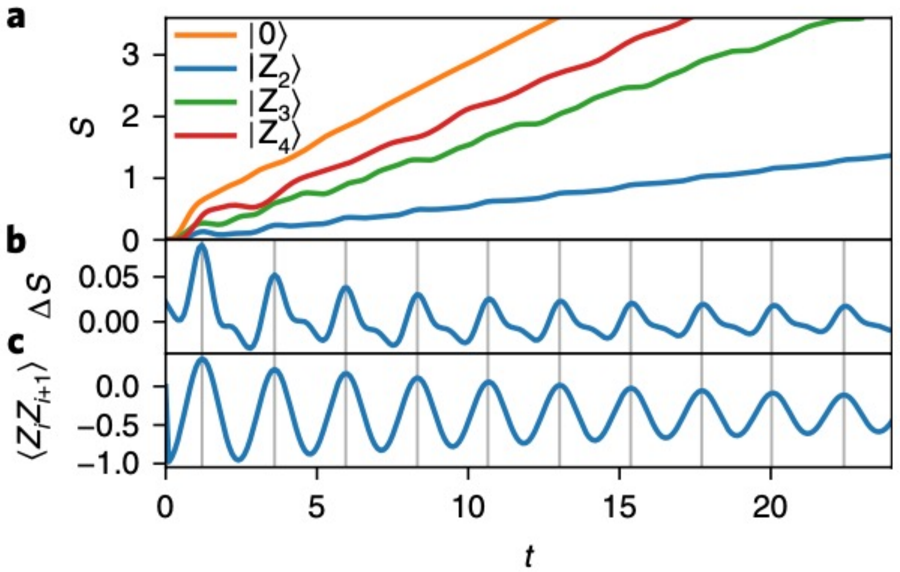
Time evolution:
$\mathrm{e}^{-\mathrm{i}H_{\mathrm{PXP}}t}$

$|0\rangle = |\cdots \circ\circ\circ\circ\circ\circ\circ\circ\circ\cdots\rangle$
$|\mathbb{Z}_2\rangle = |\cdots \bullet\circ\bullet\circ\bullet\circ\bullet\circ\cdots\rangle$
$|\mathbb{Z}_3\rangle = |\cdots \bullet\circ\circ\bullet\circ\circ\bullet\circ\circ\cdots\rangle$
$|\mathbb{Z}_4\rangle = |\cdots \bullet\circ\circ\circ\bullet\circ\circ\circ\cdots\rangle$



Revivals measured using return probability to initial state for different $|\mathbb{Z}_d\rangle$ for $L = 24$ with PBC.

- $|\mathbb{Z}_2\rangle$, $|\mathbb{Z}_3\rangle$, $|\mathbb{Z}_4\rangle$ states have revivals in an exponentially large Hilbert space.

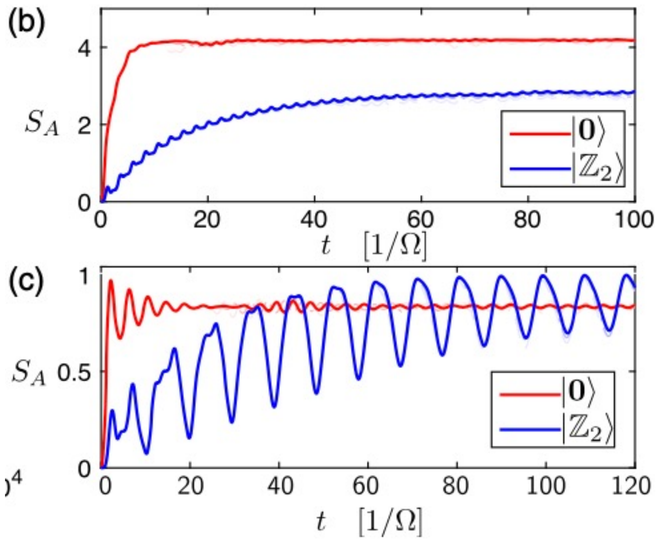


Entanglement entropy with midpoint bipartition for different initial states.

- $|\mathbb{Z}_2\rangle$, $|\mathbb{Z}_3\rangle$, $|\mathbb{Z}_4\rangle$ states have slower growth of entanglement entropy than $|0\rangle$.
- Initial state $|\mathbb{Z}_2\rangle$ leads to oscillations about the linear growth in entanglement entropy and nearest neighbor correlations.

arxiv:1711.03528, arxiv:1806.10933

Semiclassical treatment of PXP Hamiltonian using Matrix Product States (MPS)

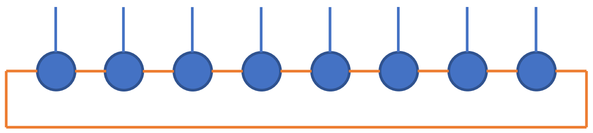


- Entanglement entropy of a 1-atom and 6-atom subsystems depend on the initial state.
- Entanglement has oscillatory behavior.
- $\mathbb{Z}_2$ initial state leads to low entanglement (approximately area law)

Matrix Product State

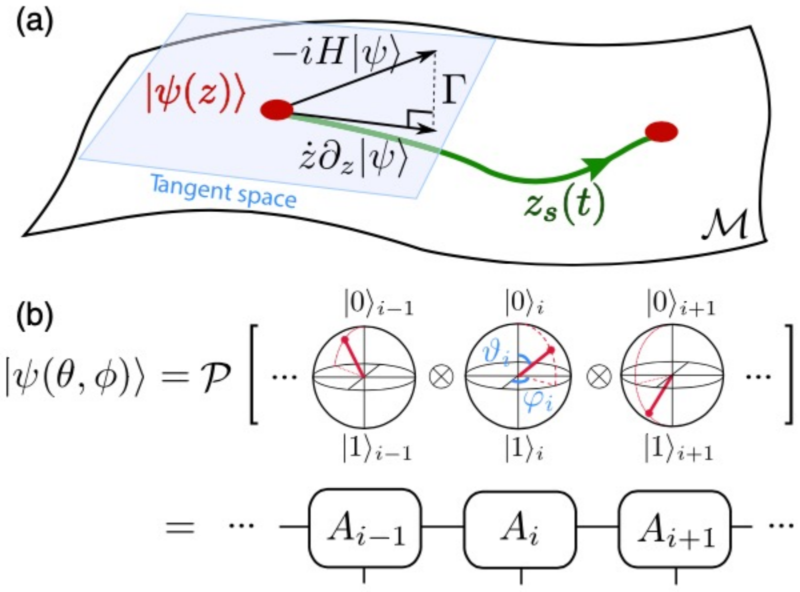
- Factorize the amplitudes as matrix of higher rank tensors
- Introduce auxiliary or virtual indices
- States of the PXP Hamiltonian Hilbert Space can be written as bond-dimension 2 MPS.
- MPS are parameterized by θ and ϕ at each site.

$$\left|\psi(\vec{\theta},\vec{\phi}\right\rangle = \mathrm{Tr}\left(A_1(\theta_1,\phi_1), A_2(\theta_2,\phi_2)\cdots A_L(\theta_L,\phi_L)\right)$$



arxiv:1807.01815, arxiv:1905.08564

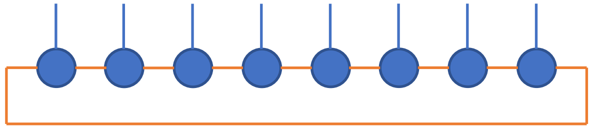
Semiclassical treatment of PXP Hamiltonian using Matrix Product States (MPS)



Time Dependent Variational Principle (TDVP)

- Variationally optimize the projection of a quantum state on a desired manifold.
- Here the manifold is that of bond-dimension 2 MPS with two-site cell parameterized by θ_{even}, θ_{odd}

- Bond dimension 2 MPS
- Assume each spin evolves in $x-z$ plane, with $\phi = 0$
- 2-site translational invariance, $\theta_{\ell+2} = \theta_\ell$
- Equations for motion for θ_{even}, θ_{odd} using the Time Dependent Variational Principle (TDVP)
- Non-linear equations => phase space picture for θ_{even}, θ_{odd}

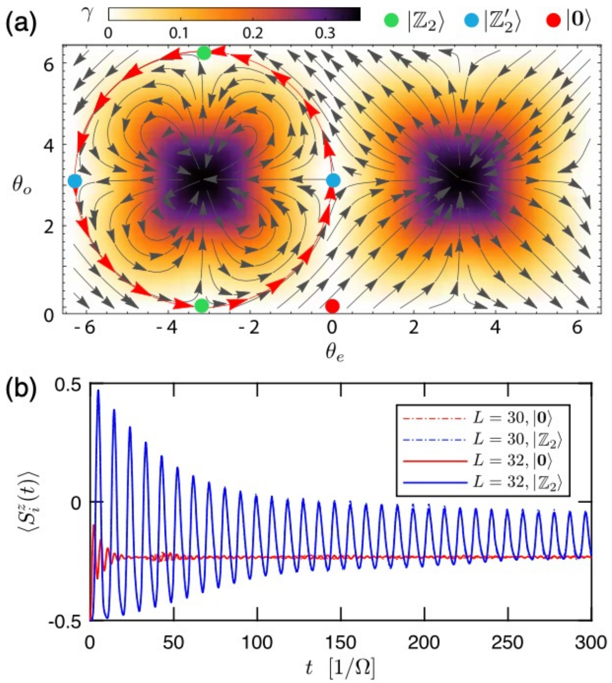


$$|\psi(\theta_{\text{even}},\theta_{\text{odd}})\rangle = \mathrm{Tr}\left(\cdots A_{\text{even}}(\theta_{\text{even}})A_{\text{odd}}(\theta_{\text{odd}})\cdots\right)$$

$$\dot{\theta}_{\text{even}} = f(\theta_{\text{even}},\theta_{\text{odd}})$$
$$\dot{\theta}_{\text{odd}} = f(\theta_{\text{odd}},\theta_{\text{even}})$$

arxiv:1807.01815, arxiv:1905.08564

Semiclassical treatment of PXP Hamiltonian using Matrix Product States (MPS)

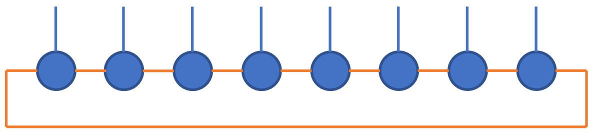


- Suggestive of mixed phase space.
- Color represents rate of leakage γ out of the MPS manifold.

Flow diagrams of $\dot{\theta}_{\text{even}}$, $\dot{\theta}_{\text{odd}}$ for different initial conditions.

- Unstable periodic orbit between states $|\mathbb{Z}_2\rangle$ and $|\mathbb{Z}_2'\rangle$.
- Motion from state $|0\rangle$ proceeds towards a saddle point.
- Similarity with persistent oscillations for $|\mathbb{Z}_2\rangle$ and equilibration for $|0\rangle$.

- Bond-dimension 2 MPS
- Assume each spin evolves in $x - z$ plane, with $\phi = 0$
- 2-site translational invariance, $\theta_{\ell+2} = \theta_\ell$
- Equations for motion for θ_{even}, θ_{odd} using the Time Dependent Variational Principle (TDVP)
- Non-linear equations => phase space picture for θ_{even}, θ_{odd}.
- Leakage out of MPS manifold is quantified.



$$|\psi(\theta_{\text{even}}, \theta_{\text{odd}})\rangle = \text{Tr}\left(\cdots A_{\text{even}}(\theta_{\text{even}}) A_{\text{odd}}(\theta_{\text{odd}}) \cdots\right)$$

$$\dot{\theta}_{\text{even}} = f(\theta_{\text{even}}, \theta_{\text{odd}})$$

$$\dot{\theta}_{\text{odd}} = f(\theta_{\text{odd}}, \theta_{\text{even}})$$

arxiv:1807.01815, arxiv:1905.08564

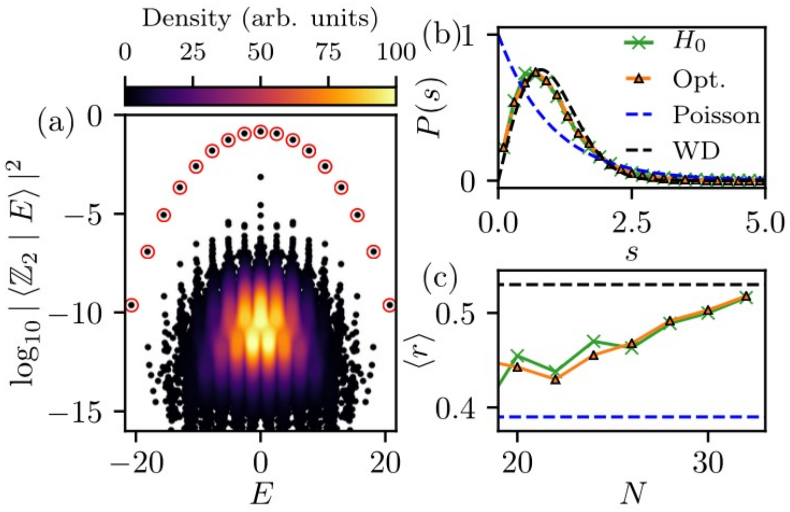
Deforming the PXP Hamiltonian

$$H_{\mathrm{PXP}} \rightarrow H_{\mathrm{PXP}} + \delta H_R$$

$$\delta H_R = -\sum_j \sum_{d=2}^{R} h_d \mathcal{P} X_j \mathcal{P} (Z_{j-2} + Z_{j+d})$$

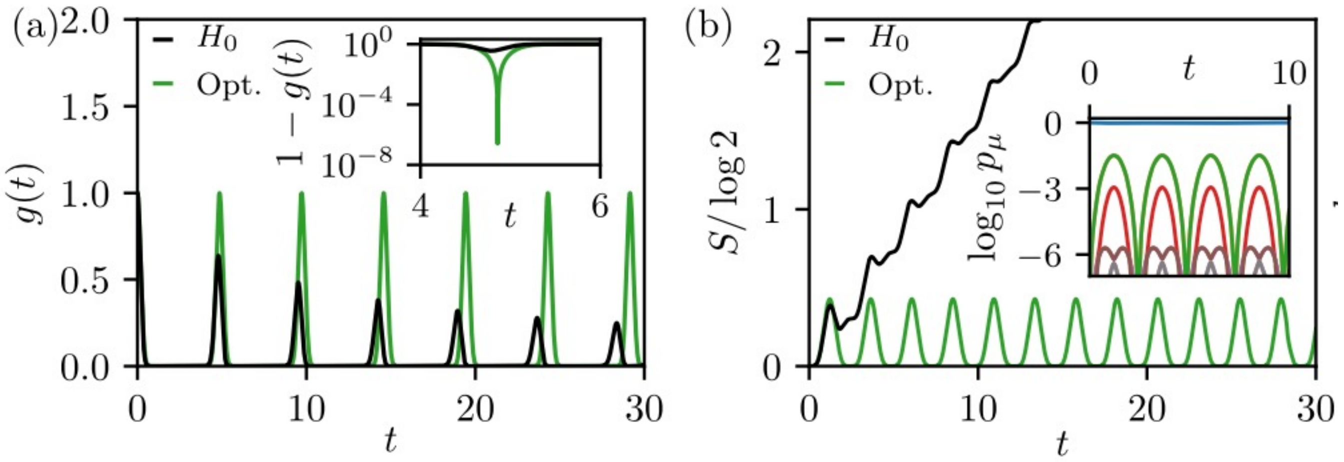
$$\mathcal{P} = \bigotimes_j \left(\frac{1}{4}(\mathbb{1} + Z_j)(\mathbb{1} + Z_{j+1}) \right)$$

Global projector with nearest neighbor constraint



PXP Hamiltonian can be deformed with long-range interactions to obtain

- Stronger revivals.
- Entanglement entropy oscillations
- More separation of “special eigenstates” from bulk spectrum



Deformed PXP Hamiltonian is still chaotic

- Energy level spacing distribution approaches Wigner Dyson for GOE as system size increases

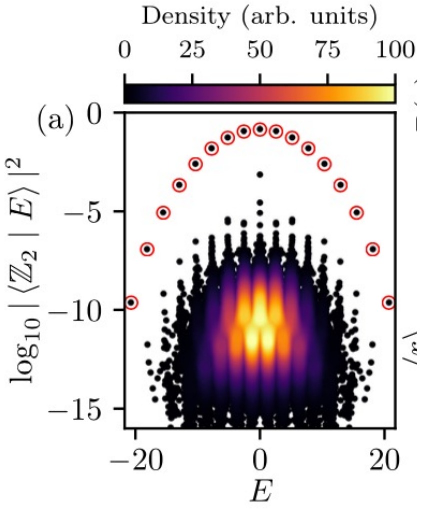
arxiv:1812.05561

Deforming the PXP Hamiltonian – Emergent SU(2)

$$H_{\mathrm{PXP}} \to H_{\mathrm{PXP}} + \delta H_R$$

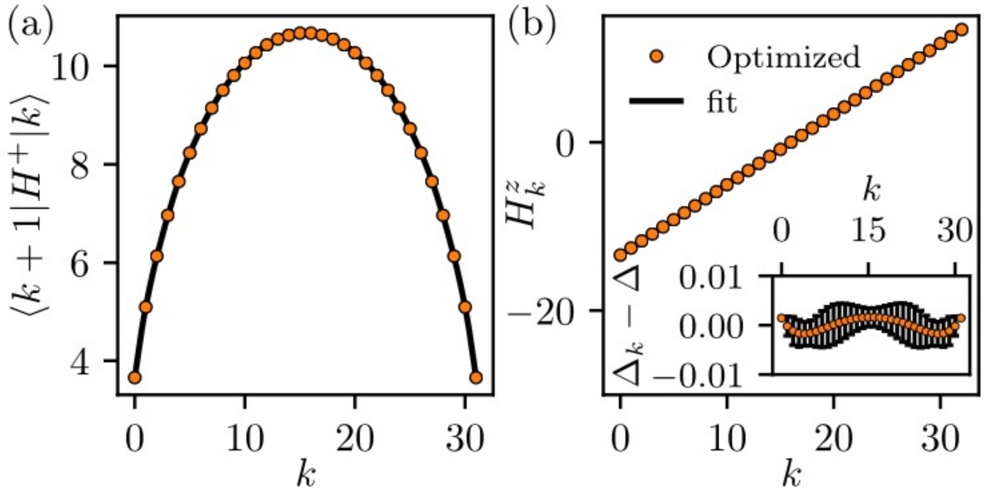
$$\delta H_R = -\sum_j \sum_{d=2}^{R} h_d \mathcal{P} X_j \mathcal{P} (Z_{j-2} + Z_{j+d})$$

$$\mathcal{P} = \bigotimes_j \left(\frac{1}{4} (\mathbb{1} + Z_j)(\mathbb{1} + Z_{j+1}) \right)$$



Projector on to "special band"

$$\mathcal{P}_{\mathcal{K}} \left[H^z, H^{\pm} \right] \mathcal{P}_{\mathcal{K}} \approx \pm \Delta \mathcal{P}_{\mathcal{K}} H^{\pm} \mathcal{P}_{\mathcal{K}}$$



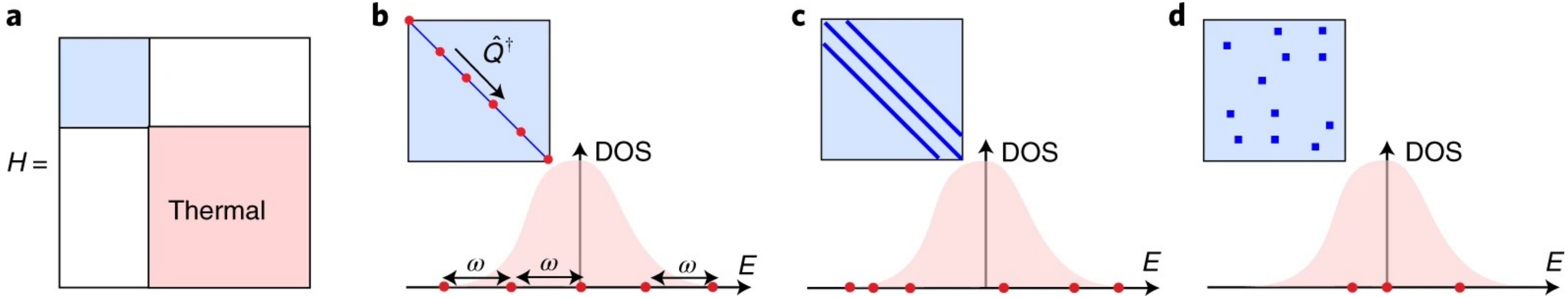
Emergent SU(2) structure

- Left: matrix elements of H^+ is approaches those of a representation of the spin raising operator, S^+.
- Right: matrix elements of H^z is approaches those of a representation of the spin operator, S^z.

arxiv:1812.05561

Mechanism of weak ergodicity breaking

$$H = H_{\mathrm{scar}} \oplus H_{\mathrm{thermal}}$$



Approximate block diagonal structure that is not related to symmetries of the Hamiltonian.

Spectrum generating algebra

- Operator Q recursively generates the eigenstates of the Hamiltonian, like a ladder operator.

Krylov restricted thermalization

- Hamiltonian generates dynamics in a restricted subspace.
- Time evolution operator has a tridiagonal structure in the scar subspace

Projector embedded scarred Hamiltonian

- Construct scarred Hamiltonian by embedding projectors to create the approximate block-diagonal structure.

arxiv:2011.09486

Summary and Other Topics

Summary

- Quantum many body scars lead to weak breaking of ergodicity in a chaotic Hamiltonian.
- Scarred Hamiltonian have a special energy band which have anomalous properties
  - Low entanglement entropy,
  - High overlaps with specific states
  - High revival probability during quench dynamics.
- Semi classical treatment using Matrix Product States and Time Dependent Variational Principle suggests a possible mixed phase space.

Other topics

- Weak violation of ETH for matrix elements in the special energy band.
- Construction of scarred Hamiltonians from well known models.
- Deformations of scarred Hamiltonians and possible integrability.
- Robustness of quantum many body scars to some perturbations.
- Hamiltonians with general fracturing of Hilbert space.
- Beyond 2 level local Hilbert Space.

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