Unambiguous Quantum State Discrimination

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(Dated: 13 Dec 2005)

We review the use of POVM's as an error-free method to distinguish the members of a finite set of quantum states, but with a finite probability of an inconclusive result. This is only possible for a set of linearly independent states. Minimization of the probability of an inconclusive result is discussed for a number of cases, including the case of two states with arbitrary a priori probabilities and the case of N symmetric states with equal a priori probabilities.

INTRODUCTION

The ability to distinguish a given quantum state from a set of possible states is of fundamental importance in quantum information processing. The properties of quantum states are such that they make available communications protocols that are theoretically superior to those possible classically. But the same properties that make quantum states such powerful resources also make some basic operations impossible or very difficult, namely state determination [1].

Exactly determining the state of an arbitrary quantum system with complete confidence is impossible. This is the cornerstone of the famous no-cloning theorem of Wooters and Zurek [2]. Research in the past two decades has shed light on how well a given state can be determined. It turns out that, while it is impossible to determine arbitrary states, it is possible to discriminate a finite set of states without making errors, as long as an inconclusive result is considered acceptable [1].

This paper will work through a simple illustrative example of how a well-chosen POVM can be used to distinguish states without ever misidentifying any, as well as look at strategies to choose POVM's such that the probability of an inconclusive result occurring is minimized.

USING POVM'S TO DISCRIMINATE NON-ORTHOGONAL STATES

POVM's are one of the basic tools of quantum information theory. They are a specific form of general measurement, but more general than projective measurements [3]. Specifically, they are not bound by the condition $P_i P_j = \delta_{ij} P_i$ that applies to projectors, though the elements of a POVM, $\{E_i\}$, must satisfy the conditions that they be positive and complete:

$$\langle\psi|E_i|\psi\rangle \geq 0 \tag{1}$$

$$\sum_i E_i = \hat{\mathbb{1}} \tag{2}$$

One scenario in which it is possible to distinguish all of the given states is that of a set of mutually orthogonal states. In that case, one can simply choose one-dimensional projectors as the POVM elements. Only the projector (e.g. $|\psi_i\rangle\langle\psi_i|$) that corresponds to the state (e.g. $|\psi_i\rangle$) will yield a measurement result.

To illustrate this, suppose we have three states, $|0\rangle$, $|1\rangle$, and $|2\rangle$, such that $\langle i|j\rangle = \delta_{ij}$. If we choose the following projectors as the POVM elements, $E_0 = |0\rangle\langle 0|$, $E_1 = |1\rangle\langle 1|$, $E_2 = |2\rangle\langle 2|$, we will get the following measurement statistics:

$$p_{E_i}(|j\rangle) = \langle j|E_i|j\rangle = \langle j|i\rangle\langle i|j\rangle = \delta_{ij} \tag{3}$$

From these probabilities, it is easy to see that only measuring with the POVM element that corresponds to the given state will yield a result. In this case it is trivial to unambiguously distinguish the states.

If, on the other hand, the states in the set are non-orthogonal, the states cannot be perfectly distinguished without error or uncertainty[1]. Suppose we have the following two states,

$$|\psi_1\rangle = |0\rangle \tag{4}$$

$$|\psi_2\rangle = a|0\rangle + b|1\rangle \tag{5}$$

where, for $i, j = 0, 1$, $\langle i|j\rangle = \delta_{ij}$, and $|a|^2 + |b|^2 = 1$. If we were able to distinguish these states reliably, we could choose POVM elements $\{E_1, E_2\}$, such that $\sum_i E_i = \hat{\mathbb{1}}$, and we would know both of the following:

$$\langle\psi_1|E_1|\psi_1\rangle = \langle\psi_2|E_2|\psi_2\rangle = 1 \tag{6}$$

$$\langle\psi_1|E_2|\psi_1\rangle = \langle\psi_2|E_1|\psi_2\rangle = 0 \tag{7}$$

The completeness of the set of POVM elements results in eq. 7, which implies that

$$\sqrt{E_2}|\psi_1\rangle = \sqrt{E_1}|\psi_2\rangle = 0 \tag{8}$$

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[1] Adapted from Nielsen and Chuang [3], p.87.

Here we've used the fact that all E_i are positive, which allows us to write $E_i = \sqrt{E_i}\sqrt{E_i}$. Now let us take a closer look at $\langle\psi_2|E_1|\psi_2\rangle$:

$$\begin{aligned}\langle\psi_2|E_1|\psi_2\rangle &= (a^*\langle 0| + b^*\langle 1|)E_1(a|0\rangle + b|1\rangle) \quad (9)\\ &= |a|^2\langle 0|E_1|0\rangle + a^*b\langle 0|E_1|1\rangle \\ &\quad + ab^*\langle 1|E_1|0\rangle + |b|^2\langle 1|E_1|1\rangle \quad (10)\\ &= |a|^2 + a^*b\langle 0|\sqrt{E_1}\sqrt{E_1}|1\rangle \\ &\quad + ab^*\langle 1|\sqrt{E_1}\sqrt{E_1}|0\rangle + 0 \quad (11)\\ &= |a|^2 + 0 + 0 + 0 = |a|^2 \\ &= 0\end{aligned}$$

Where we have utilized eq. (8). This means for $\langle\psi_2|E_1|\psi_2\rangle = 0$, we must have $a = 0 \Leftrightarrow |\psi_2\rangle = |1\rangle$. What we have shown is that the assumption of distinguishability requires our states to be orthogonal.

DISTINGUISHING NON-ORTHOGONAL STATES WITHOUT ERROR

While it is not possible to always distinguish states without error, it turns out that it is possible to distinguish the states some portion of the time without error, if the possibility of an inconclusive outcome is allowed. This procedure was first discovered by Ivanovic [4] and later more fully explored by Dieks [5] and Peres [6]. In this scenario, for N states there are N+1 outcomes: one certain outcome for each of the N states and one outcome that tells us that we do not know the state. We will first look at how this is possible with just two states in a two dimensional system.

A simple example of two states

We will now look at a simple example of a POVM which yields an error-free distinction of two non-orthogonal states, but also includes an inconclusive result[2]. Suppose we are given the following states:

$$|\psi_\alpha\rangle = |0\rangle \quad (12)$$

$$|\psi_\beta\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad (13)$$

We can see that it would be impossible to distinguish these states perfectly using von Neumann measurements. By choosing the right POVM elements we can distinguish the two without error. Consider the following three element POVM:

$$E_1 = \frac{\sqrt{2}}{1+\sqrt{2}}|1\rangle\langle 1| \quad (14)$$

$$E_2 = \frac{\sqrt{2}}{1+\sqrt{2}}\frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2} \quad (15)$$

$$E_3 = \hat{\mathbb{1}} - E_1 - E_2 \quad (16)$$

$$E_3 = \frac{\sqrt{2}}{1+\sqrt{2}}\left[\frac{(\sqrt{2}+1)|0\rangle\langle 0| + (\sqrt{2}-1)|1\rangle\langle 1| + |0\rangle\langle 1|}{2} + \frac{|1\rangle\langle 0|}{2}\right] \quad (17)$$

We can see that our POVM elements satisfy the necessary completeness relation: $\sum_i E_i = \hat{\mathbb{1}}$. Note that none of these elements are orthogonal, as $E_iE_j \neq \delta_{ij}$ for distinct i, j.

Now lets look at the probabilities of the possible outcomes. If we have state $|\psi_\alpha\rangle$, we will get results E_1, E_2, and E_3 with the following probabilities:

$$p_{E_1}(|\psi_\alpha\rangle) = \langle\psi_\alpha|E_1|\psi_\alpha\rangle = 0 \quad (18)$$

$$p_{E_2}(|\psi_\alpha\rangle) = \langle\psi_\alpha|E_2|\psi_\alpha\rangle = \frac{1}{2}(\frac{\sqrt{2}}{1+\sqrt{2}}) \quad (19)$$

$$p_{E_3}(|\psi_\alpha\rangle) = \langle\psi_\alpha|E_3|\psi_\alpha\rangle = 1 - \frac{1}{2}(\frac{\sqrt{2}}{1+\sqrt{2}}) = \frac{1}{\sqrt{2}} \quad (20)$$

From this, we can be certain that if we get result E_1 we don't have $|\psi_\alpha\rangle$. Now let's look at the measurement statistics for state $|\psi_\beta\rangle$.

$$p_{E_1}(|\psi_\beta\rangle) = \langle\psi_\beta|E_1|\psi_\beta\rangle = \frac{1}{2}(\frac{\sqrt{2}}{1+\sqrt{2}}) \quad (21)$$

$$p_{E_2}(|\psi_\beta\rangle) = \langle\psi_\beta|E_2|\psi_\beta\rangle = 0 \quad (22)$$

$$p_{E_3}(|\psi_\beta\rangle) = \langle\psi_\beta|E_3|\psi_\beta\rangle = 1 - \frac{1}{2}(\frac{\sqrt{2}}{1+\sqrt{2}}) = \frac{1}{\sqrt{2}} \quad (23)$$

Analogous to the previous case, we can be certain that the state is not $|\psi_\beta\rangle$ if the result is E_2. In this case, regardless of which state we have, the probability of an inconclusive result is $P_I = \frac{1}{\sqrt{2}} \approx 0.71$.

Geometrically, we can think of the POVM elements as projector-like operators, corresponding to states, $|\psi_i^\perp\rangle$,

[2] Adapted from Nielsen and Chuang [3], p.92.

orthogonal to the states in the set, $|\psi_i\rangle$. For our two-dimensional example, we chose state vectors that lie perpendicular to the state vector in question. Since the second state is not the same as the first (and assuming it is non-zero), it must have some component in the perpendicular direction of the first. A result that corresponds to a projection into the perpendicular direction (to the first state vector) means the state in hand must be the second state ($|\psi_\beta\rangle$ in the example). In other words,

$$E_1 \propto |\psi_\alpha^\perp\rangle\langle\psi_\alpha^\perp| = |1\rangle\langle 1| \tag{24}$$

was chosen because $\langle\psi_\alpha^\perp|\psi_\alpha\rangle = \langle 1|0\rangle = 0$. Similarly

$$\langle\psi_\beta^\perp|\psi_\beta\rangle = (\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}})(\langle 0| - \langle 1|)(|0\rangle + |1\rangle) = 0 \tag{25}$$

and

$$E_2 \propto |\psi_\beta^\perp\rangle\langle\psi_\beta^\perp| = \frac{1}{2}(\langle 0| - \langle 1|)(|0\rangle - |1\rangle) \tag{26}$$

Another approach to two non-orthogonal states

In their 1996 paper, Huttner et al. [7] described a general procedure for two states, $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$, where $|\langle\psi_\alpha|\psi_\beta\rangle| = \cos\gamma$, and γ is the angle between the state vectors in two dimensions. Their procedure involves extending the two dimensional Hilbert space to three dimensions. This is accomplished by using a third state, $|\phi_0\rangle$, which is orthogonal to both $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$. They further choose two more states, $|\phi_1\rangle$ and $|\phi_2\rangle$, such that $\langle\phi_i|\phi_j\rangle = \delta_{ij}$, for $i, j = 0, 1, 2$, forming a three dimension orthonormal basis. Using a unitary evolution operator, U, they rotate the state off the original plane by angle θ about a vector

$$|v\rangle \equiv \frac{1}{\sqrt{2}}(|\phi_1\rangle - |\phi_2\rangle) \tag{27}$$

If θ is chosen such that $\cos\theta = \tan\frac{\gamma}{2}$, the original states $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ are transformed (with some simplification) to

$$|\psi'_\alpha\rangle = U|\psi_\alpha\rangle = \sqrt{2}\sin\frac{\gamma}{2}|\phi_1\rangle + \sqrt{\cos\gamma}|\phi_0\rangle \tag{28}$$

$$|\psi'_\beta\rangle = U|\psi_\beta\rangle = \sqrt{2}\sin\frac{\gamma}{2}|\phi_2\rangle + \sqrt{\cos\gamma}|\phi_0\rangle \tag{29}$$

We can see that $|\psi'_\alpha\rangle$ and $|\psi'_\beta\rangle$ are orthogonal to $|\phi_2\rangle$ and $|\phi_1\rangle$ respectively[3]. By choosing POVM elements

$$E_1 = |\phi_1\rangle\langle\phi_1| \tag{30}$$

$$E_2 = |\phi_2\rangle\langle\phi_2| \tag{31}$$

we can easily distinguish the initial states, $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$. Of course we are also bound by the completeness relation, $\sum_i E_i = \hat{\mathbb{1}}$, so we must have

$$E_3 = \hat{\mathbb{1}} - E_1 - E_2 = |\phi_0\rangle\langle\phi_0| \tag{32}$$

which corresponds to an inconclusive result. We see that the measurement statistics are

$$p_{E_1}(|\psi_\alpha\rangle) = \langle\psi'_\alpha|E_1|\psi'_\alpha\rangle = 2\sin^2\frac{\gamma}{2} \tag{33}$$

$$p_{E_2}(|\psi_\alpha\rangle) = \langle\psi'_\alpha|E_2|\psi'_\alpha\rangle = 0 \tag{34}$$

$$p_{E_3}(|\psi_\alpha\rangle) = \langle\psi'_\alpha|E_3|\psi'_\alpha\rangle = \cos\gamma \tag{35}$$

and

$$p_{E_1}(|\psi_\beta\rangle) = \langle\psi'_\beta|E_2|\psi'_\beta\rangle = 0 \tag{36}$$

$$p_{E_2}(|\psi_\beta\rangle) = \langle\psi'_\beta|E_1|\psi'_\beta\rangle = 2\sin^2\frac{\gamma}{2} \tag{37}$$

$$p_{E_3}(|\psi_\beta\rangle) = \langle\psi'_\beta|E_3|\psi'_\beta\rangle = \cos\gamma \tag{38}$$

Noting that

$$2\sin^2\frac{\gamma}{2} + \cos\gamma = 2\sin^2\frac{\gamma}{2} + (1 - 2\sin^2\frac{\gamma}{2}) = 1$$

we see that the POVM is complete. In effect, Huttner et al. have mapped two non-orthogonal states to two orthogonal-like states (they can be distinguished with projectors), albeit in three dimensions. Here the probability of an inconclusive result is $p_I = \cos\gamma$, which is also $|\langle\psi_\alpha|\psi_\beta\rangle|$. We see that orthogonal states ($\gamma = \frac{\pi}{2}$) will be perfectly distinguishable with $p_I = 0$.

CONDITIONS FOR ERROR-FREE STATE DISCRIMINATION

In 1998, Chefles [8] showed that error-free discrimination of states is only possible if the states form a linearly independent set. He used the following proof. We first assume that we have N distinguishable states, $|\psi_i\rangle$. Our POVM will then have N elements, E_i, to distinguish each of the N states, and one element, E_I, which corresponds to the inconclusive result. For the N elements to distinguish the states, we must have

$$\langle\psi_i|E_j|\psi_i\rangle = p_i\delta_{ij} \tag{39}$$

Without assuming the states are linearly dependent, we can express them as

$$|\psi_i\rangle = \sum_k c_{ik}|\psi_k\rangle \tag{40}$$

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[3] See the Huttner et al. paper [7] for a good graphical depiction of their procedure.

If we now plug this back into eq. (39), we get

$$\langle\psi_i|E_j|\psi_i\rangle = \sum_{k,k'} c^*_{ik'}c_{ik}\langle\psi_{k'}|E_j|\psi_k\rangle = p_i\delta_{ij} \qquad (41)$$

To simplify $\langle\psi_{k'}|E_j|\psi_k\rangle$, we can use the Cauchy-Schwartz inequality:

$$|\langle\psi_{k'}|E_j|\psi_k\rangle|^2 \leqslant \langle\psi_{k'}|E_j|\psi_{k'}\rangle\langle\psi_k|E_j|\psi_k\rangle = p_{k'}\delta_{k'j}p_k\delta_{kj} \qquad (42)$$

Using this with eq. (39) we get

$$\langle\psi_{k'}|E_j|\psi_k\rangle = p_k\delta_{kk'}\delta_{kj} \qquad (43)$$

Inserting this into eq. (41), we get

$$\sum_k |c_{ik}|^2\langle\psi_k|E_j|\psi_k\rangle = p_i\delta_{ij} \qquad (44)$$

From eq. (39) we can conclude that we must have $|c_{ik}|^2 = \delta_{ik}$. Looking at eq. (40), we see this conditions implies that the states are linearly independent of one another, if they are to be distinguishable.

MINIMIZING THE PROBABILITY OF AN INCONCLUSIVE RESULT

While avoiding the mis-identification of states is very useful, it is also desireable to minimize the probability of inconclusive results for a given system. In this section we will look at the work reported by different authors on minimizing the probability of an inconclusive result, p_I, for different cases.

Two states

The early work of Ivanovic [4], Dieks [5], and Peres [6] looked at the discrimination of two states with equal a priori probabilities, $p_1 = p_2 = \frac{1}{2}$, as in the above examples. They found the probability of an inconclusive result to be

$$p_I = |\langle\psi_1|\psi_2\rangle| \qquad (45)$$

which again shows that fully reliable discrimination only occurs when the states are orthogonal.

In their 1995 paper [9], Jaeger and Shimnoy looked at two states with arbitrary a priori probabilities, $p_1 = 1 - p_2$. Following a procedure similar to Huttner et al., they found that the minimum probability of an inconclusive result for two states, p_I, was

$$p_I^{min} = \frac{1}{2} - \frac{1}{2}\sqrt{(1 - 4p_1p_2|\langle\psi_1|\psi_2\rangle|^2)} \qquad (46)$$

Three states

In 1998 Peres and Terno [10] published results on minimizing p_I for three linearly independent states in three dimensions with arbitrary a priori probabilities. Following a procedure with similarities to the above procedure by Huttner et al., Peres and Terno found the coefficients, k_i, of the POVM elements

$$E_i = k_i|\phi_i\rangle\langle\phi_i| \qquad (47)$$

which are constrained by the fact that

$$E_I = \hat{1\!\!1} - \sum_{i=1}^{3} E_i \qquad (48)$$

must be positive. For p_I^{min}, the coefficients, k_i should be as large as possible. They found that these coefficients lie on a convex surface in the first octant of three space, though they did not give an analytical expression for p_I^{min}.

N symmetric states with equal a priori probabilities

In their 1998 paper [10], Peres and Terno also laid out a strategy to achieve p_I^{min} for N states. In the same year, Chefles and Barnett [11] were able to analytically determine the necessary POVM corresponding to p_I^{min} for N symmetric states with equal a priori probabilities.

A set of states is symmetric if there is a a unitary transformation operator, U, such that

$$|\psi_i\rangle = U|\psi_{i-1}\rangle = U^i|\psi_0\rangle \qquad (49)$$

$$|\psi_0\rangle = U|\psi_{N-1}\rangle \qquad (50)$$

$$U^N = \hat{1\!\!1} \qquad (51)$$

For equal a priori probabilities, they found that the minimum probability for an inconclusive result was

$$p_I^{min} \geqslant 1 - N \times min(|c_r|^2) \qquad (52)$$

where N is the number of states and c_r comes from a state expansion

$$|\psi_i\rangle = \sum_{k=0}^{N-1} c_k e^{\frac{2\pi ijk}{N}}|\phi_k\rangle \qquad (53)$$

which has the reciprocal (orthogonal) state

$$|\psi_i^{\perp}\rangle = Z^{-\frac{1}{2}} \sum_{r=0}^{N-1} c_r^{*-1} e^{\frac{2\pi ijr}{N}}|\phi_r\rangle \qquad (54)$$

where $Z = \sum_r |c_r|^{-2}$ and (finally)

$$|c_r|^2 = \frac{1}{N^2} \sum_{i,i'} e^{\frac{-2\pi i r(j-j')}{N}} \langle \phi_{j'} | \phi_j \rangle \qquad (55)$$

Once again it can be seen that if the states are all orthogonal, and thus $|c_r|^2 = \frac{N}{N^2} = N^{-1}$, we will have $p_I = 0$ and the states will be fully distinguishable.

CONCLUSION

We have reviewed the use of POVM's as a method to distinguish quantum states from a finite set. This provides us with an error-free way to tell even non-orthogonal states apart, as long as an inconclusive result is acceptable. For certain situations, physicists have been able to find the minimum possible probability of an inconclusive result and/or the POVM elements which will yield the minimum probability.

ACKNOWLEDGEMENTS

This paper was written as part of Carlton Caves' quantum information theory course at the University of New Mexico (Fall 2005). For helpful comments and discussion, I would like to thank Carlton Caves and Sergio Boixo.

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