7 Nonparametric Methods

SW Section 7.11 and 9.4-9.5

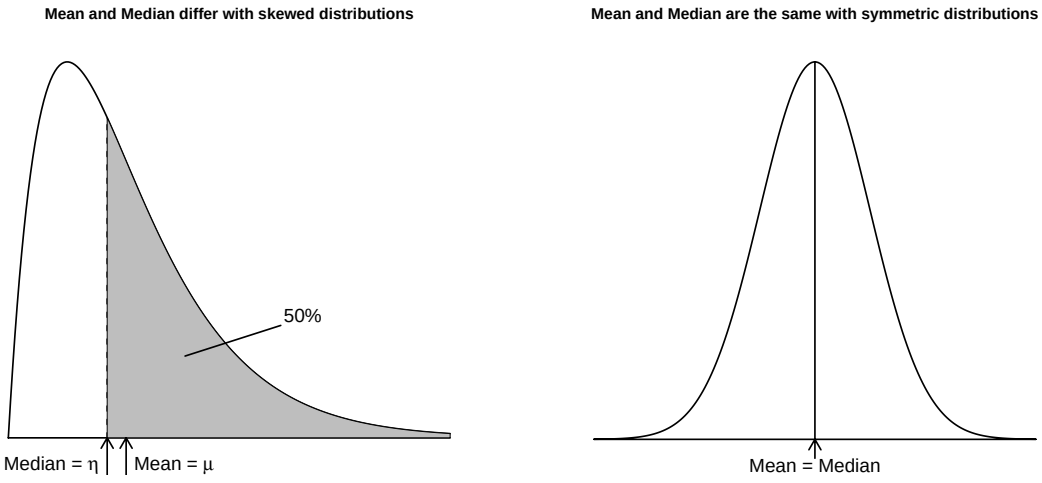
Nonparametric methods do not require the normality assumption of classical techniques. I will describe and illustrate selected **non-parametric methods**, and compare them with classical methods. Some motivation and discussion of the strengths and weaknesses of non-parametric methods is given.

The Sign Test and CI for a Population Median

The **sign test** assumes that you have a random sample from a population, but makes no assumption about the population shape. The standard $t-$test provides inferences on a population mean. The sign test, in contrast, provides inferences about a **population median**.

If the population frequency curve is symmetric (see below), then the population median, identified by η, and the population mean μ are identical. In this case the sign procedures provide inferences for the population mean.

The idea behind the sign test is straightforward. Suppose you have a sample of size m from the population, and you wish to test $H_0 : \eta = \eta_0$ (a given value). Let S be the number of sampled observations *above* η_0. If H_0 is true, you expect S to be approximately one-half the sample size, $.5m$. If S is much greater than $.5m$, the data suggests that $\eta > \eta_0$. If S is much less than $.5m$, the data suggests that $\eta < \eta_0$.



S has a **Binomial distribution** when H_0 is true. The Binomial distribution is used to construct a test with size α (approximately). For a two-sided alternative $H_A : \eta \neq \eta_0$, the test rejects H_0 when S is significantly different from $.5m$, as determined from the reference Binomial distribution. One sided tests use the corresponding lower or upper tail of the distribution. To generate a CI for η, you can exploit the duality between CI and tests. A $100(1-\alpha)\%$ CI for η consists of all values η_0 not rejected by a two-sided size alpha test of $H_0 : \eta = \eta_0$.

Comments:

1. Minitab omits all observations at exactly η_0, so m is the sample size after omissions. This should not be much of a concern unless the measurements are coarsely rounded.

2. Not all test sizes and confidence levels are possible because the test statistic S is discrete valued. Minitab gives an exact p-value for the test, and approximates the desired confidence level using a non-linear interpolation algorithm.

3. To implement the sign procedures in Minitab follow: `Stat > Nonparametrics > 1-Sample Sign`. The dialog box allows you to specify a test or a CI, but not both at the same time. The tests can be two-sided or one-sided.

4. Only two-sided CIs are available, so you have to be clever to get a one-sided bound. For example, to get an upper 95% bound, you take the upper limit from a 90% two-sided confidence interval. The rational for this is that with the 90% two-sided CI, the population parameter will fall above the upper limit 5% of the time and fall below the lower limit 5% of the time. Thus, you are 95% confident that the population parameter falls below the upper limit of this interval, which gives us our one-sided bound. The same logic applies if you want to generalize the one-sided confidence bounds to arbitrary confidence levels and to lower one-sided bounds - always double the error rate of the desired one-sided bound to get the error rate of the required two-sided interval! For example, if you want a lower 99% bound (with a 1% error rate), use the lower limit on the 98% two-sided CI (which has a 2% error rate).

Example: Income Data

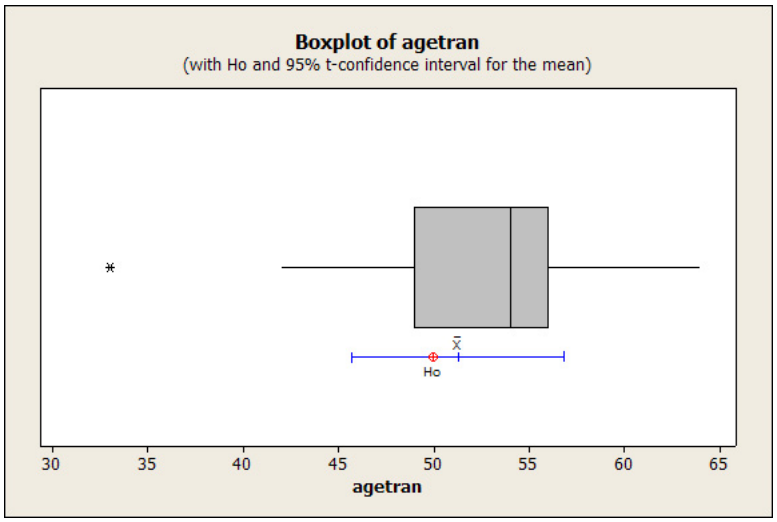
Recall that the income distribution is extremely skewed, with two extreme outliers at 46 and 1110. The presence of the outliers has a dramatic effect on the 95% CI for the population mean income μ, which goes from -101 to 303 (in 1000 dollar units). This $t-$CI is suspect because the normality assumption is unreasonable. A CI for the population median income η is more sensible because the median is likely to be a more reasonable measure of typical value. Using the sign procedure, you are 95% confident that the population median income is between 2.79 and 10.95 (times 1000 dollars).

```
Data Display

Income
      7   1110      7      5      8     12      0      5      2      2     46
      7

One-Sample T: Income

Variable   N     Mean    StDev  SE Mean          95% CI
Income    12  100.917  318.008   91.801  (-101.136, 302.969)


Sign CI: Income
```

```
Sign confidence interval for median

                                 Confidence
                      Achieved     Interval
         N  Median  Confidence  Lower  Upper  Position
Income  12    7.00      0.8540   5.00   8.00         4
                        0.9500   2.79  10.95       NLI
                        0.9614   2.00  12.00         3

**** REMARK: NLI stands for non-linear interpolation
```

Example: Age at First Heart Transplant

Recall that the distribution of ages is skewed to the left with a lower outlier. A question of interest is whether the "typical age" at first transplant is 50. This can be formulated as a test about the population median η or as a test about the population mean μ, depending on the interpretation. The sign test for $H_0 : \eta = 50$ against $H_A : \eta \neq 50$ has a p-value of .549, which is not sufficient to reject H_0. A 95% CI for η is 48.42 to 56.16 years, which includes the hypothesized median age of 50. Similar conclusions are reached with the $t-$CI and the test on μ, but you should have less confidence in these results because the normality assumption is tenuous. You could check normality, using a normal scores test.




```
Data Display

agetran
   33   42   49   49   51   54   54   54   56   58   64

Stem-and-Leaf Display: agetran

Stem-and-leaf of agetran  N  = 11
Leaf Unit = 1.0

 1   3  3
 1   3
 2   4  2
 4   4  99
(4)  5  1444
 3   5  68
 1   6  4
```

```
Descriptive Statistics: agetran

Variable   N  N*   Mean  SE Mean  StDev  Minimum     Q1  Median     Q3  Maximum
agetran   11   0  51.27     2.49   8.26    33.00  49.00   54.00  56.00    64.00

One-Sample T: agetran

Test of mu = 50 vs not = 50


Variable   N     Mean   StDev  SE Mean         95% CI           T      P
agetran   11  51.2727  8.2594   2.4903  (45.7240, 56.8215)  0.51  0.620


Sign Test for Median: agetran

Sign test of median =  50.00 versus not = 50.00

          N  Below  Equal  Above       P  Median
agetran  11      4      0      7  0.5488   54.00

Sign CI: agetran

Sign confidence interval for median

                              Confidence
                   Achieved     Interval
          N  Median Confidence  Lower  Upper  Position
agetran  11   54.00     0.9346  49.00  56.00         3
                    0.9500  48.42  56.16       NLI
                    0.9883  42.00  58.00         2
```

Wilcoxon Signed Rank Procedures

The **Wilcoxon** procedure assumes you have a random sample from a population with a symmetric frequency curve. The curve need not be normal. The test and CI can be viewed as procedures for either the population median or mean.

To illustrate the computation of the Wilcoxon statistic W, suppose you wish to test $H_0 : \mu = \mu_0 = 10$ on the data below. The test statistic requires us to compute the **signs** of $X_i - \mu_0$ and the **ranks** of $|X_i - \mu_0|$. Ties in $|X_i - \mu_0|$ get the average rank and observations at μ_0 (here 10) are always discarded. The Wilcoxon statistic is the **sum of the signed ranks for observations above** $\mu_0 = 10$. For us

$$W = 6 + 4.5 + 8 + 2 + 4.5 + 7 = 32.$$

X_i	$X_i - 10$	sign	$\lvert X_i - 10\rvert$	rank	sign*rank
20	10	+	10	6	6
18	8	+	8	4.5	4.5
23	13	+	13	8	8
5	-5	-	5	3	-3
14	4	+	4	2	2
8	-2	-	2	1	-1
18	8	+	8	4.5	4.5
22	12	+	12	7	7

The sum of all ranks is always $.5m(m+1)$, where m is the sample size. If H_0 is true, you expect W to be approximately $.5 * .5m(m+1) = .25m(m+1)$. Why? Recall that W adds up the ranks for observations above μ_0. If H_0 is true, you expect 1/2 of all observations to be above μ_0, assuming the population distribution is symmetric. The ranks of observations above μ_0 should add to approximately 1/2 times the sum of all ranks. You reject H_0 in favor of $H_A : \mu \neq \mu_0$ if W is much larger than, or much smaller than $.25m(m+1)$. One sided tests can also be constructed. The Wilcoxon CI for μ is computed in a manner analogous to that described for the sign CI.

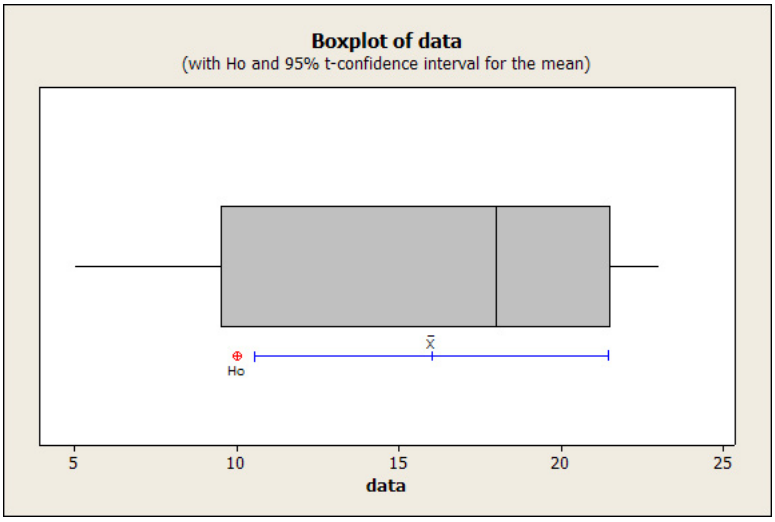
Here, $m = 8$ so the sum of all ranks is $.5 * 8 * 9 = 36$ (check yourself). The expected value of W is $.5 * .5 * 8 * 9 = 18$. Is the observed value of W **far from** the expected value? To formally answer this question, we need to use the Wilcoxon procedures, which are implemented in Minitab by following the sequence: `Stat > Nonparametrics > 1-Sample Wilcoxon`.

Example: Play Data

The boxplot indicates that the distribution is fairly symmetric, so the Wilcoxon method is reasonable (so is a t-CI and test). The p-value for testing $H_0 : \mu = 10$ against a two-sided alternative is .059. This would not lead to rejecting H_0 at the 5% level.

Although I asked for a 95% CI, I got a 94.1% CI. The W test statistic is discrete, so not all confidence levels are achievable. Minitab gives the closest possible level. I underlined the estimated median of 16.5 given by the CI procedure. This disagrees with the median you get in the data description. The CI median is computed using Walsh averages - see the Minitab help for an explanation.

```
Data Display

data
   20   18   23    5   14    8   18   22

Descriptive Statistics: Income

Variable   N  N*   Mean  SE Mean  StDev  Minimum   Q1  Median    Q3  Maximum
Income    12   0  100.9     91.8  318.0      0.0  2.8     7.0  11.0   1110.0

One-Sample T: data

Test of mu = 10 vs not = 10


Variable  N     Mean   StDev  SE Mean         95% CI            T      P
data      8  16.0000  6.5247   2.3068  (10.5452, 21.4548)  2.60  0.035


Wilcoxon Signed Rank Test: data

Test of median = 10.00 versus median not = 10.00

            N
           for   Wilcoxon         Estimated
      N  Test  Statistic      P     Median
data  8     8       32.0  0.059      16.50


Wilcoxon Signed Rank CI: data

                                 Confidence
       Estimated    Achieved     Interval
      N    Median  Confidence  Lower  Upper
data  8      16.5        94.1   11.0   21.0
             ----
```



Nonparametric Analyses of Paired Data

Nonparametric methods for single samples can be used to analyze paired data because the difference between responses within pairs is the unit of analysis.

Example: Sleep Remedies

I will illustrate Wilcoxon methods on the paired comparison of two remedies A and B for insomnia. The number of hours of sleep gained on each method was recorded. Unlike the parametric paired $t-$test, you must create the sample of differences to do the non-parametric analysis in Minitab.

The boxplot shows that distribution of differences is reasonably symmetric but not normal. Recall that the Shapiro-Wilk test of normality was significant at the 5% level (p-value=.035). It is sensible to use the Wilcoxon procedure on the differences. Let μ_B be the population mean sleep gain on remedy B, and μ_A be the population mean sleep gain on remedy A. You are 94.7% confident that $\mu_B - \mu_A$ is between 0.8 and 2.7 hours. Putting this another way, you are 94.7% confident that μ_B exceeds μ_A by between 0.8 and 2.7 hours. The p-value for testing $H_0 : \mu_B - \mu_A = 0$ against a two-sided alternative is .008, which strongly suggests that $\mu_B \neq \mu_A$. This agrees with the CI. Note that the t-CI and test give qualitatively similar conclusions as the Wilcoxon methods, but the $t-$test p-value is about twice as large.

If you are uncomfortable with the symmetry assumption, you could use the sign CI for the population median difference between B and A. I will note that a 95% CI for the median difference goes from 0.93 to 2.01 hours.

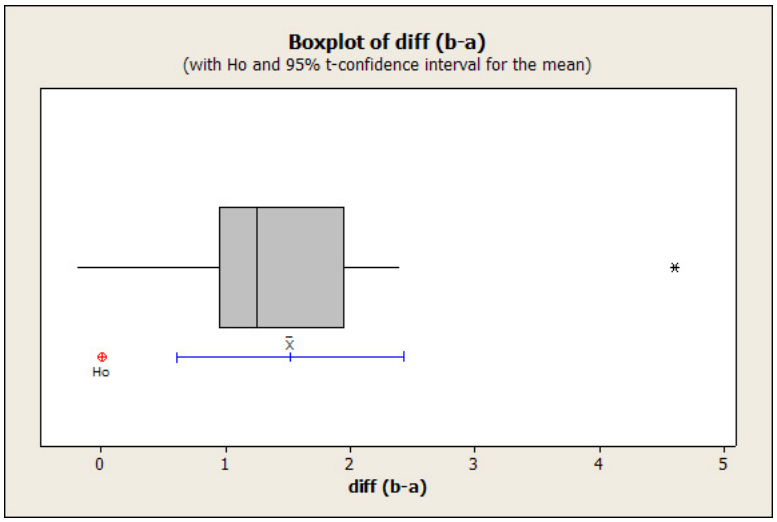
```
Data Display

                  diff
Row     a     b  (b-a)
  1   0.7   1.9   1.2
  2  -1.6   0.8   2.4
  3  -0.2   1.1   1.3
  4  -1.2   0.1   1.3
```

```
  5   0.1  -0.1   -0.2
  6   3.4   4.4    1.0
  7   3.7   5.5    1.8
  8   0.8   1.6    0.8
  9   0.0   4.6    4.6
 10   2.0   3.0    1.0
```

One-Sample T: diff (b-a)

Test of mu = 0 vs not = 0

Variable	N	Mean	StDev	SE Mean	95% CI	T	P
diff (b-a)	10	1.52000	1.27174	0.40216	(0.61025, 2.42975)	3.78	0.004



Wilcoxon Signed Rank CI: diff (b-a)

	N	Estimated Median	Achieved Confidence	Confidence Interval Lower	Upper
diff (b-a)	10	1.30	94.7	0.80	2.70

Wilcoxon Signed Rank Test: diff (b-a)

Test of median = 0.000000 versus median not = 0.000000

	N	N for Test	Wilcoxon Statistic	P	Estimated Median
diff (b-a)	10	10	54.0	0.008	1.300

Sign CI: diff (b-a)

Sign confidence interval for median

	N	Median	Achieved Confidence	Confidence Interval Lower	Upper	Position
diff (b-a)	10	1.250	0.8906	1.000	1.800	3
			0.9500	0.932	2.005	NLI
			0.9785	0.800	2.400	2

Comments on One-Sample Nonparametric Methods

For this discussion, I will assume that the underlying population distribution is (approximately) symmetric, which implies that population means and medians are equal (approximately). For symmetric distributions the t, sign, and Wilcoxon procedures are all appropriate.

If the underlying population distribution is extremely skewed, you can use the sign procedure to get a CI for the population median. Alternatively, as illustrated on HW 2, you can transform the data to a scale where the underlying distribution is nearly normal, and then use the classical $t-$methods. Moderate degrees of skewness will not likely have a big impact on the standard $t-$test and CI.

The one-sample $t-$test and CI are optimal when the underlying population frequency curve is normal. Essentially this means that the $t-$CI is, on average, narrowest among all CI procedures with given level, or that the t-test has the highest power among all tests with a given size. The width of a CI provides a measure of the sensitivity of the estimation method. For a given level CI, the narrower CI better pinpoints the unknown population mean.

With heavy-tailed symmetric distributions, the t-test and CI tend to be conservative. Thus, for example, a nominal 95% $t-$CI has actual coverage rates higher than 95%, and the nominal 5% t-test has an actual size smaller than 5%. The $t-$test and CI possess a property that is commonly called **robustness of validity**. However, data from heavy-tailed distributions can have a profound effect on the **sensitivity** of the t-test and CI. Outliers can dramatically inflate the standard error of the mean, causing the CI to be needlessly wide, and tests to have diminished power (outliers typically inflate p-values for the $t-$test). The sign and Wilcoxon procedures downweight the influence of outliers by looking at sign or signed-ranks instead of the actual data values. These two nonparametric methods are somewhat less efficient than the t-methods when the population is normal (efficiency is about .64 and .96 for the sign and Wilcoxon methods relative to the normal t-methods, where efficiency is the ratio of sample sizes needed for equal power), but can be infinitely more efficient with heavier than normal tailed distributions. In essence, the t-methods do not have a **robustness of sensitivity**.

Nonparametric methods have gained widespread acceptance in many scientific disciplines, but not all. Scientists in some disciplines continue to use classical $t-$methods because they believe that the methods are robust to non-normality. As noted above, this is a robustness of validity, not sensitivity. This misconception is unfortunate, and results in the routine use of methods that are less powerful than the non-parametric techniques. **Scientists need to be flexible and adapt their tools to the problem at hand, rather than use the same tool indiscriminately!** I have run into suspicion that use of nonparametric methods was an attempt to "cheat" in some way – properly applied, they are excellent tools that **should** be used.

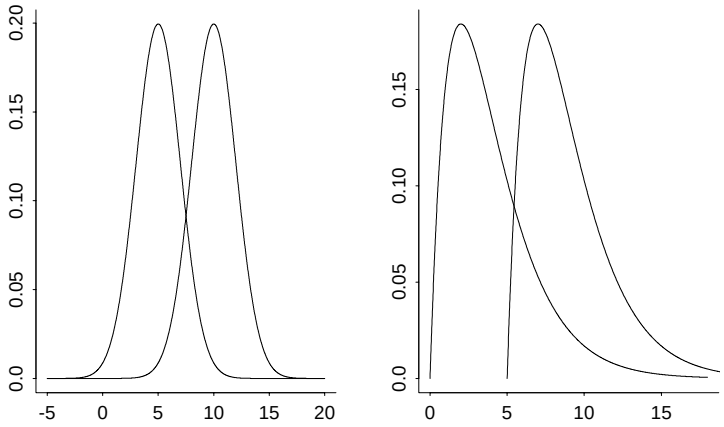
A minor weakness of nonparametric methods is that they do not easily generalize to complex modelling problems. A great deal of progress has been made in this area, but most software packages have not included the more advanced techniques.

Nonparametric statistics used to refer almost exclusively to the set of methods such as we have been discussing that provided analogs like tests and CIs to the normal theory methods without requiring the assumption of sampling from normal distributions. There is now a large area of statistics also called nonparametric methods not focused on these goals at all. In our department we have a course titled "Nonparametric Curve Estimation & Image Reconstruction", where the

focus is much more general than relaxing an assumption of normality. In that sense, what we are covering in this course could be considered "classical" nonparametrics.

(Wilcoxon-) Mann-Whitney Two Sample Procedure

The WMW procedure assumes you have independent random samples from the two populations, and assumes that the populations have the same shapes and spreads (the frequency curves for the two populations are "shifted" versions of each other - see below). The frequency curves are not required to be symmetric. The WMW procedures give a CI and tests on the difference $\eta_1 - \eta_2$ between the two population medians. If the populations are symmetric, then the methods apply to $\mu_1 - \mu_2$.



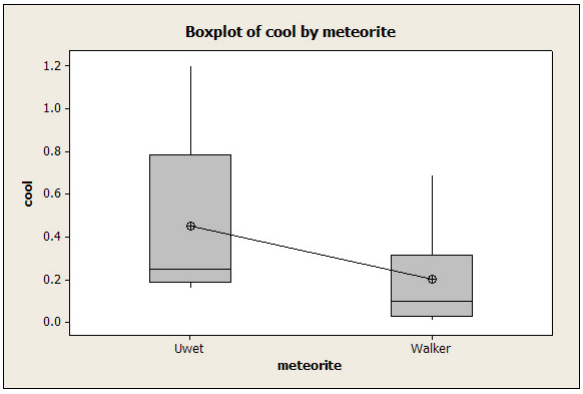
The Minitab on-line help explains the exact WMW procedure actually calculated. I will discuss a very good approximation to the exact method that is easier to understand. The WMW procedure is based on ranks. The two samples are combined, ranked from smallest to largest (1=smallest) and separated back into the original samples. If the two populations have equal medians, you expect the average rank in the two samples to be roughly equal. The WMW test computes a classical two sample t-test using the pooled variance on the ranks to assess whether the sample mean ranks are significantly different.

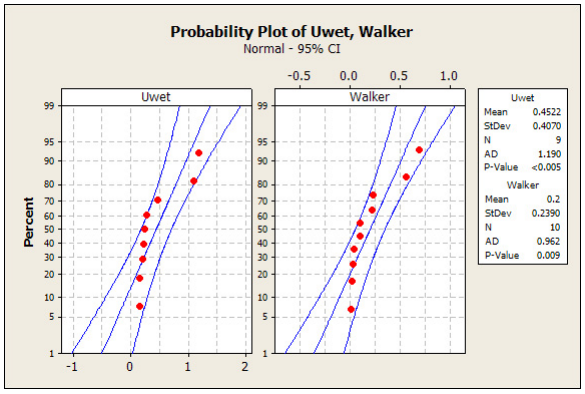
The WMW test and CI are implemented in Minitab by following these steps:
`Stat > Nonparametrics > Mann-Whitney`. The data must be UNSTACKED. The test and CI are generated simultaneously. A two-sided CI is given, even if a one-sided test is requested.

Example: Comparison of Cooling Rates of Walker and Uwet Meteorites.

The Uwet and Walker Co. data were read into two columns of the Minitab worksheet. A primary interest is comparing the population "typical" cooling rate measurements.

```
Data Display

Row  Uwet  Walker  cool  meteorite  ranks
  1  0.21    0.69  0.21  Uwet         9.0
  2  0.25    0.23  0.25  Uwet        13.0
  3  0.16    0.10  0.16  Uwet         7.5
  4  0.23    0.03  0.23  Uwet        11.5
  5  0.47    0.56  0.47  Uwet        15.0
  6  1.20    0.10  1.20  Uwet        19.0
  7  0.29    0.01  0.29  Uwet        14.0
  8  1.10    0.02  1.10  Uwet        18.0
  9  0.16    0.04  0.16  Uwet         7.5
 10          0.22  0.69  Walker      17.0
 11                0.23  Walker      11.5
 12                0.10  Walker       5.5
 13                0.03  Walker       3.0
 14                0.56  Walker      16.0
 15                0.10  Walker       5.5
 16                0.01  Walker       1.0
 17                0.02  Walker       2.0
 18                0.04  Walker       4.0
 19                0.22  Walker      10.0

Descriptive Statistics: cool

Variable  meteorite   N  N*    Mean  SE Mean   StDev  Minimum      Q1  Median
cool      Uwet        9   0   0.452    0.136   0.407    0.160   0.185   0.250
          Walker     10   0  0.2000   0.0756  0.2390   0.0100  0.0275  0.1000

Variable  meteorite      Q3  Maximum
cool      Uwet        0.785    1.200
          Walker     0.3125   0.6900
```





The boxplots, stem and leaf, and normal scores plots show that the distributions are rather skewed to the right. Both the AD and RJ tests of normality indicate that a normality assumption is unreasonable for each population.

I carried out the standard two-sample procedures to see what happens. The pooled variance and Satterthwaithe results are comparable, which is expected because the sample standard deviations and sample sizes are roughly equal. Both tests indicate that the mean cooling rates for Uwet and Walker Co. meteorites are not significantly different at the 10% level. You are 95% confident that the mean cooling rate for Uwet is at most .1 less, and no more than .6 greater than that for Walker. (in degrees per million years).

```
Two-Sample T-Test and CI: cool, meteorite
```

```
Two-sample T for cool

meteorite   N   Mean  StDev  SE Mean
Uwet        9  0.452  0.407     0.14
Walker     10  0.200  0.239    0.076


Difference = mu (Uwet) - mu (Walker)
Estimate for difference:  0.252222
95% CI for difference:  (-0.066627, 0.571071)
T-Test of difference = 0 (vs not =): T-Value = 1.67  P-Value = 0.113  DF = 17
Both use Pooled StDev = 0.3289


Two-Sample T-Test and CI: cool, meteorite

Two-sample T for cool

meteorite   N   Mean  StDev  SE Mean
Uwet        9  0.452  0.407     0.14
Walker     10  0.200  0.239    0.076


Difference = mu (Uwet) - mu (Walker)
Estimate for difference:  0.252222
95% CI for difference:  (-0.086133, 0.590578)
T-Test of difference = 0 (vs not =): T-Value = 1.62  P-Value = 0.130  DF = 12
```

Given the marked skewness, a nonparametric procedure is more appropriate. The Wilcoxon-Mann-Whitney comparison of population medians is reasonable. Why? The WMW test of equal population medians is significant (barely) at the 5% level. You are 95% confident that median cooling rate for Uwet exceeds that for Walker by between 0+ and .45 degrees per million years.

The difference between the WMW and t-test p-values and CI lengths (i.e. the WMW CI is narrower and the p-value smaller) reflects the effect of the outliers on the sensitivity of the standard tests and CI.

In the worksheet, I computed the ranks by first stacking the data, then following `Data > Rank`. I conducted a two-sample t-test on ranks to show you that the p-value is close to the WMW p-value, as expected.

```
Descriptive Statistics: ranks <- describe ranks, to think about WMW results

Variable  meteorite   N  N*   Mean  SE Mean  StDev  Minimum    Q1  Median
ranks     Uwet        9   0  12.72     1.41   4.24     7.50  8.25   13.00
          Walker     10   0   7.55     1.82   5.75     1.00  2.75    5.50

Variable  meteorite     Q3  Maximum
ranks     Uwet       16.50    19.00
          Walker     12.63    17.00

Two-Sample T-Test and CI: ranks, meteorite

Two-sample T for ranks

meteorite   N   Mean  StDev  SE Mean
Uwet        9  12.72   4.24      1.4
Walker     10   7.55   5.75      1.8


Difference = mu (Uwet) - mu (Walker)
```

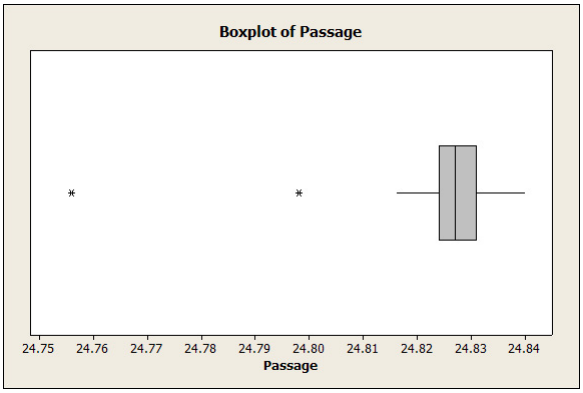
```
Estimate for difference:  5.17222
95% CI for difference:  (0.23049, 10.11395)  <<<<<---- NOT Interpretable
T-Test of difference = 0 (vs not =): T-Value = 2.21  P-Value = 0.041  DF = 17
Both use Pooled StDev = 5.0978


Mann-Whitney Test and CI: Uwet, Walker

          N  Median
Uwet      9  0.2500
Walker   10  0.1000


Point estimate for ETA1-ETA2 is 0.1750
95.5 Percent CI for ETA1-ETA2 is (-0.0002,0.4501) <<<<<--- Can't get exact 95% CI
W = 114.5
Test of ETA1 = ETA2 vs ETA1 not = ETA2 is significant at 0.0500
The test is significant at 0.0497 (adjusted for ties)
```

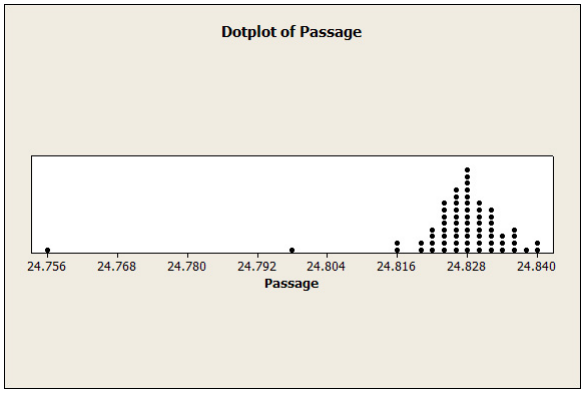
Example: Newcombe's Data

Experiments of historical importance were performed beginning in the eighteenth century to determine such physical constants as the mean density of the earth, the distance from the earth to the sun, and the velocity of light. An interesting series of experiments to determine the velocity of light was begun in 1875. The first method used, and reused with refinements several times thereafter, was the rotating mirror method. In this method a beam of light is reflected off a rapidly rotating mirror to a fixed mirror at a carefully measured distance from the source. The returning light is re-reflected from the rotating mirror at a different angle, because the mirror has turned slightly during the passage of the corresponding light pulses. From the speed of rotation of the mirror and from careful measurements of the angular difference between the outward-bound and returning light beams, the passage time of light can be calculated for the given distance. After averaging several calculations and applying various corrections, the experimenter can combine mean passage time and distance for a determination of the velocity of light. Simon Newcombe, a distinguished American scientist, used this method during the year 1882 to generate the passage time measurements given below, in microseconds. The travel path for this experiment was 3721 meters in length, extending from Ft. Meyer, on the west bank of the Potomac River in Washington, D.C. to a fixed mirror at the base of the Washington Monument.

The problem is to determine a 95% CI for the "true" passage time, which is taken to be the mean of the population of measurements that were or could have been taken by this experiment.

```
Data Display

Passage
   24.828   24.827   24.824   24.831   24.836   24.837   24.836   24.827
   24.826   24.839   24.829   24.826   24.816   24.821   24.819   24.828
   24.825   24.826   24.827   24.833   24.828   24.827   24.833   24.840
   24.825   24.824   24.825   24.828   24.830   24.828   24.826   24.824
   24.828   24.824   24.798   24.830   24.820   24.821   24.826   24.822
   24.827   24.832   24.825   24.829   24.834   24.829   24.823   24.836
   24.828   24.830   24.836   24.831   24.832   24.832   24.816   24.756
   24.822   24.829   24.832   24.829   24.832   24.823   24.827   24.824
   24.825   24.823
```





The data set is skewed to the left, due to the presence of two extreme outliers that could potentially be misrecorded observations. Without additional information I would be hesitant to apply normal theory methods (the t-test), even though the sample size is "large". Basically, folklore says you can apply standard normal theory methods in large samples. This is true, but how large the sample must be depends on how skewed, or heavy tailed, the underlying population distribution is. Furthermore, the t-test still suffers from a lack of robustness of sensitivity, even in large samples. A formal normal scores test (not provided) would reject, at the 0.01 level, the normality assumption needed for the standard methods.

The table below gives 95% t, sign and Wilcoxon CIs. I am more comfortable with the sign CI for the population median than the Wilcoxon method, which assumes symmetry. The question asks for CI for a population mean, but this is probably because the book I got this problem from was illustrating methods for means!

Method	Limits
t	24.8236 - 24.8289
sign	24.8260 - 24.8284
Wilcoxon	24.8260 - 24.8285

Note the big difference between the nonparametric and the t-CI. The nonparametric CIs are about 1/2 as wide as the t-CI. This reflects the impact that outliers have on the standard deviation, which directly influences the CI width.

Computation note: Minitab is pretty poor at formatting output for data like this with the nonparametric procedures. When it printed the lower and upper bounds for the sign and Wilcoxon CIs, it reported both upper and lower bounds of 24.83 – a fairly useless report. The problem is the number of digits recorded in the original data. To get the values above I subtracted 24.7 from the original values, calculated CIs, and added 24.7 back to the CI limits. SAS tends to print a ridiculous number of digits; Minitab usually makes prettier output, but they should be more careful here.

Alternative Analyses for ANOVA and Planned Comparisons

The classical ANOVA assumes that the populations have normal frequency curves and the populations have equal variances (or spreads). You learned formal tests for these assumptions earlier. When the assumptions do not hold, you can try one of the following two approaches. Before describing alternative methods, I will note that deviations from normality in one or more samples might be expected in a comparison involving many samples. You should downplay small deviations from normality in problems involving many samples.

Kruskal-Wallis ANOVA

The **Kruskal-Wallis** (KW) test is a non-parametric method for testing the hypothesis of equal population medians against the alternative that not all population medians are equal. The procedure assumes you have independent random samples from populations with frequency curves having identical shapes and spreads.

The KW ANOVA is essentially the standard ANOVA based on ranked data. That is, we combine the samples, rank the observations from smallest to largest, and then return the ranks to the original samples and do the standard ANOVA using the ranks.

The KW ANOVA is a multiple sample analog of the Wilcoxon-Mann-Whitney two sample procedure. Hence, multiple comparisons for a KW analysis, be they FSD or Bonferroni comparisons, are based on the two sample WMW procedure.

Transforming Data

The distributions in many data sets are skewed to the right with outliers. If the sample spreads, say s and IQR, increase with an increasing mean or median, you can often **transform data** to a scale where the normality and the constant spread assumption are more nearly satisfied. The transformed data are analyzed using the standard ANOVA. The two most commonly used transforms for this problem are the square root and natural logarithm, provided the data are non-negative. I will give you some idea why this might work in class.

If the original distributions are nearly symmetric, but heavy tailed, non-linear transformations will tend to destroy the symmetry. Many statisticians recommend methods based on trimmed means for such data. These methods are not commonly used by other researchers.

Example: Hydrocarbon (HC) Emissions Data

These data are the HC emissions at idling speed, in ppm, for automobiles of different years of manufacture. The data are a random sample of all automobiles tested at an Albuquerque shopping center. (It looks like we need to find some newer cars!)

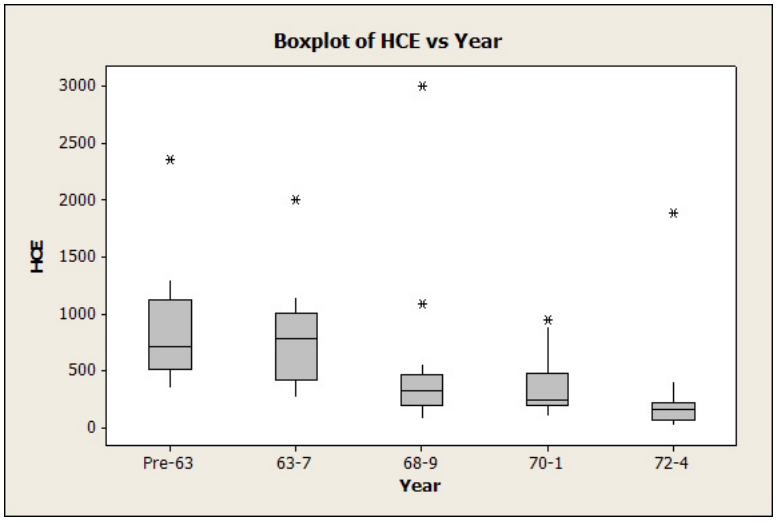
The standard ANOVA shows significant differences among the mean HC emissions. However, the standard ANOVA is inappropriate because the distributions are extremely skewed to the right due to presence of outliers in each sample.

```
Data Display
```

Row	Pre-63	63-7	68-9	70-1	72-4
1	2351	620	1088	141	140
2	1293	940	388	359	160
3	541	350	111	247	20
4	1058	700	558	940	20
5	411	1150	294	882	223
6	570	2000	211	494	60
7	800	823	460	306	20
8	630	1058	470	200	95
9	905	423	353	100	360
10	347	900	71	300	70
11		405	241	223	220
12		780	2999	190	400
13		270	199	140	217
14			188	880	58
15			353	200	235
16			117	223	1880
17				188	200
18				435	175
19				940	85
20				241	

Descriptive Statistics: Pre-63, 63-7, 68-9, 70-1, 72-4

Variable	N	N*	Mean	SE Mean	StDev	Minimum	Q1	Median	Q3	Maximum
Pre-63	10	0	891	187	592	347	509	715	1117	2351
63-7	13	0	801	126	455	270	414	780	999	2000
68-9	16	0	506	177	708	71	191	324	468	2999
70-1	20	0	381.5	64.4	287.9	100.0	192.5	244.0	479.3	940.0
72-4	19	0	244.1	94.2	410.8	20.0	60.0	160.0	223.0	1880.0



One-way ANOVA: Pre-63, 63-7, 68-9, 70-1, 72-4

Source	DF	SS	MS	F	P
Factor	4	4226834	1056709	4.34	0.003
Error	73	17759968	243287		
Total	77	21986802			

S = 493.2 R-Sq = 19.22% R-Sq(adj) = 14.80%

```
                                Individual 95% CIs For Mean Based on
                                Pooled StDev
Level    N   Mean  StDev  ---------+---------+---------+---------+
Pre-63  10  890.6  591.6                    (----------*---------)
63-7    13  801.5  454.9                  (--------*--------)
68-9    16  506.3  707.8         (-------*-------)
70-1    20  381.5  287.9      (-------*------)
72-4    19  244.1  410.8  (------*-------)
                           ---------+---------+---------+---------+
                                  300       600       900      1200

Pooled StDev = 493.2
```

The boxplots show that the typical HC emissions appear to increase as the age of car increases (the simplest description). Although the spread in the samples, as measured by the IQR, also increases as age increases, I am more comfortable with the KW ANOVA, in part because the KW analysis is not too sensitive to differences in spreads among samples. This point is elaborated upon later. As described earlier, the KW ANOVA is essentially an ANOVA based on the ranks. I give below the ANOVA based on ranks and the output from the KW procedure. They give similar p-values, and lead to the conclusion that there are significant differences among the population median HC emissions. A simple description is that the population median emission tends to increase with the age of the car. You should follow up this analysis with Mann-Whitney multiple comparisons.

The KW ANOVA is conducted in Minitab by following these steps: `Stat > Nonparametrics > Kruskal-Wallis`. The data must be STACKED, and you need to specify the response variable, and the factor that defines the groups.

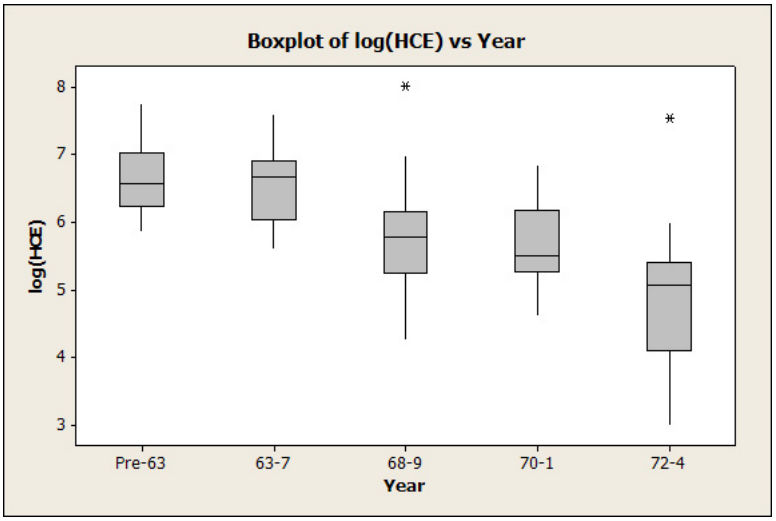
```
One-way ANOVA: hce_rank versus Year

Source  DF     SS    MS      F      P
Year     4  16329  4082  12.85  0.000
Error   73  23200   318
Total   77  39529


Kruskal-Wallis Test: HCE versus Year

Kruskal-Wallis Test on HCE

Year      N  Median  Ave Rank      Z
63-7     13   780.0      58.6   3.33
68-9     16   323.5      37.4  -0.41
70-1     20   244.0      36.6  -0.67
72-4     19   160.0      20.3  -4.25
Pre-63   10   715.0      60.4   3.12
Overall  78               39.5

H = 31.80  DF = 4  P = 0.000
H = 31.81  DF = 4  P = 0.000  (adjusted for ties)
```

It is common to transform the data to a log scale when the spread increases as the median or mean increases. The data have already been STACKED, so it is straightforward to transform the

HCE levels to the log scale using the Minitab calculator. Side-by-side boxplots of the transformed data are given on the same page as the boxplots of the untransformed data.

After transformation, the samples have roughly the same spread (IQR and s) and shape. The transformation does not completely eliminate the outliers. However, I am more comfortable with a standard ANOVA on this scale than with the original data. A difficulty here is that the ANOVA is comparing population mean log HC emission. Summaries for the ANOVA on the log hydrocarbon emissions levels are given below.

```
One-way ANOVA: hce_rank versus Year

Source  DF     SS    MS      F      P
Year     4  16329  4082  12.85  0.000
Error   73  23200   318
Total   77  39529

S = 17.83   R-Sq = 41.31%   R-Sq(adj) = 38.09%

                                Individual 95% CIs For Mean Based on
                                Pooled StDev
Level    N   Mean  StDev  --+---------+---------+---------+-------
63-7    13  58.58  12.67                        (------*------)
68-9    16  37.44  21.16             (-----*-----)
70-1    20  36.58  18.82             (----*-----)
72-4    19  20.29  19.08   (-----*----)
Pre-63  10  60.35  11.89                                (------*-------)
                          --+---------+---------+---------+-------
                            15        30        45        60

Pooled StDev = 17.83
```



The boxplot of the log-transformed data reinforces the reasonableness of the original KW analysis. Why? The log-transformed distributions have fairly similar shapes and spreads, so a KW analysis on these data is sensible. The ranks for the original and log-transformed data are identical, so the KW analyses on the log-transformed data and the original data must lead to the same conclusions. This suggests that the KW ANOVA is not overly sensitive to differences in spreads among the samples.

There are two reasonable analyses here: the standard ANOVA using log HC emissions, and the KW analysis of the original data. The first analysis gives a comparison of mean log-HC emissions. The second involves a comparison of median HC emissions. A statistician would present both analyses to the scientist who collected the data to make a decision on which was more meaningful (independently of the results!). Multiple comparisons would be performed relative to the selected analysis.

Example: Hodgkin's Disease Study

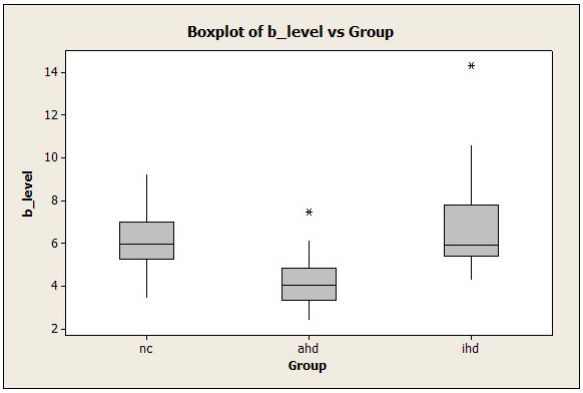
Plasma bradykininogen levels were measured in normal subjects, in patients with active Hodgkin's disease, and in patients with inactive Hodgkin's disease. The globulin bradykininogen is the precursor substance for bradykinin, which is thought to be a chemical mediator of inflammation. The data (in micrograms of bradykininogen per milliliter of plasma) are displayed below. The three samples are denoted by *nc* for normal controls, *ahd* for active Hodgkin's disease patients, and *ihd* for inactive Hodgkin's disease patients.

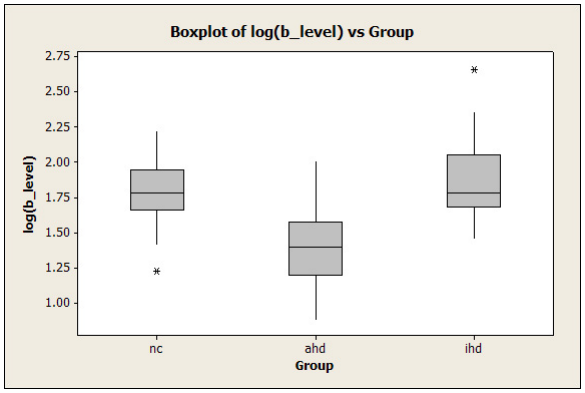
The medical investigators wanted to know if the three samples differed in their bradykininogen levels. Carry out the statistical analysis you consider to be most appropriate, and state your conclusions to this question.

```
Data Display
```

```
Data Display

Row    nc   ahd    ihd
  1  5.37  3.96   5.37
  2  5.80  3.04  10.60
  3  4.70  5.28   5.02
  4  5.70  3.40  14.30
  5  3.40  4.10   9.90
  6  8.60  3.61   4.27
  7  7.48  6.16   5.75
  8  5.77  3.22   5.03
  9  7.15  7.48   5.74
 10  6.49  3.87   7.85
 11  4.09  4.27   6.82
 12  5.94  4.05   7.90
 13  6.38  2.40   8.36
 14  9.24  5.81   5.72
 15  5.66  4.29   6.00
 16  4.53  2.77   4.75
 17  6.51  4.40   5.83
 18  7.00         7.30
 19  6.20         7.52
 20  7.04         5.32
 21  4.82         6.05
 22  6.73         5.68
 23  5.26         7.57
 24               5.68
 25               8.91
 26               5.39
 27               4.40
 28               7.13
```

```
Descriptive Statistics: nc, ahd, ihd

Variable   N  N*   Mean  SE Mean  StDev  Minimum     Q1  Median     Q3  Maximum
nc        23   0  6.081    0.284  1.362    3.400  5.260   5.940  7.000    9.240
ahd       17   0  4.242    0.316  1.303    2.400  3.310   4.050  4.840    7.480
ihd       28   0  6.791    0.411  2.176    4.270  5.375   5.915  7.780   14.300
```





Although the spread (IQR, s) in the *ihd* sample is somewhat greater than the spread in the other samples, the presence of skewness and outliers in the boxplots is a greater concern regarding the use of the classical ANOVA. The shapes and spreads in the three samples are roughly identical, so a Kruskal-Wallis nonparametric ANOVA appears ideal. As a sidelight, I transformed plasma levels to a log scale to reduce the skewness and eliminate the outliers. The boxplots of the transformed data show reasonable symmetry across groups, but outliers are still present. I will stick with the Kruskal-Wallis ANOVA (although it would not be much of a problem to use the classical ANOVA on transformed data).

Let η_{nc} = population median plasma level for normal controls, η_{ahd} = population median plasma level for active Hodgkin's disease patients, and η_{ihd} = population median plasma level for inactive Hodgkin's disease patients. The KW test of $H_0 : \eta_{nc} = \eta_{ahd} = \eta_{ihd}$ is highly significant ($p-value = .000$), suggesting differences among the population median plasma levels. The Kruskal-Wallis ANOVA summary is given below.

```
Kruskal-Wallis Test: b_level versus Group

Kruskal-Wallis Test on b_level

Group     N  Median  Ave Rank      Z
 nc      23   5.940      38.3   1.14
ahd      17   4.050      15.9  -4.47
ihd      28   5.915      42.6   2.83
Overall  68              34.5

H = 20.56  DF = 2  P = 0.000
H = 20.57  DF = 2  P = 0.000  (adjusted for ties)
```

I followed up the KW ANOVA with Bonferroni comparisons of the samples, using the Mann-Whitney two sample procedure. There are three comparisons, so an overall FER of .05 is achieved by doing the individual tests at the .05/3=.0167 level. Alternatively, you can use 98.33% CI for differences in population medians.

Remember that the WMW two-sample comparisons requires UNSTACKED data, whereas the KW required STACKED data!

```
Mann-Whitney Test and CI: nc, ahd

      N  Median
nc   23   5.940
ahd  17   4.050


Point estimate for ETA1-ETA2 is 1.910
98.4 Percent CI for ETA1-ETA2 is (0.860,2.900)
W = 605.0
Test of ETA1 = ETA2 vs ETA1 not = ETA2 is significant at 0.0003
The test is significant at 0.0003 (adjusted for ties)


Mann-Whitney Test and CI: nc, ihd

      N  Median
nc   23   5.940
ihd  28   5.915


Point estimate for ETA1-ETA2 is -0.345
98.3 Percent CI for ETA1-ETA2 is (-1.559,0.680)
W = 552.5
Test of ETA1 = ETA2 vs ETA1 not = ETA2 is significant at 0.3943
The test is significant at 0.3943 (adjusted for ties)


Mann-Whitney Test and CI: ahd, ihd

      N  Median
ahd  17   4.050
ihd  28   5.915


Point estimate for ETA1-ETA2 is -2.145
98.4 Percent CI for ETA1-ETA2 is (-3.500,-1.320)
W = 209.0
Test of ETA1 = ETA2 vs ETA1 not = ETA2 is significant at 0.0000
The test is significant at 0.0000 (adjusted for ties)
```

The only comparison with a p-value greater than .0167 involved the *nc* and *ihd* samples. The comparison leads to two groups, and is consistent with what we see in the boxplots.

```
ahd   nc   ihd
---   --------
```

You have sufficient evidence to conclude that the plasma bradykininogen levels for active Hodgkin's disease patients is lower than the population median levels for normal controls, and for patients with inactive Hodgkin's disease. You do not have sufficient evidence to conclude that the population median levels for normal controls and for patients with inactive Hodgkin's disease are different. The CIs give an indication of size of differences in the population medians.

Planned Comparisons

Bonferroni multiple comparisons are generally preferred to Fisher's least significant difference approach. Fisher's method does not control the family error rate and produces too many spurious significant differences (claims of significant differences that are due solely to chance variation and not to actual differences in population means). However, Bonferroni's method is usually very conservative when a large number of comparisons is performed - large differences in sample means are needed to claim significance. A way to reduce this conservatism is to avoid doing all possible comparisons. Instead, one should, when possible, decide a priori (before looking at the data) which comparisons are of primary interest, and then perform only those comparisons.

For example, suppose a medical study compares five new treatments with a control (a six group problem). The medical investigator may not be interested in all 15 possible comparisons, but only in which of the five treatments differ on average from the control. Rather than performing the 15 comparisons, each at the say .05/15 = .0033 level, she could examine the five comparisons of interest at the .05/5 = .01 level. By deciding beforehand which comparisons are of interest, she can justify using a .01 level for the comparisons, instead of the more conservative .0033 level needed when doing all possible comparisons.

To illustrate this idea, consider the KW analysis of HC emissions. We saw that there are significant differences among the population median HC emissions. Given that the samples have a natural ordering

Sample	Year of manufacture
1	Pre 1963
2	63 - 67
3	68 - 69
4	70 - 71
5	72 - 74

you may primarily be interested in whether the population medians for cars manufactured in consecutive samples are identical. That is, you may be primarily interested in the following 4 comparisons:

Pre 1963	vs	63 - 67
63 - 67	vs	68 - 69
68 - 69	vs	70 - 71
70 - 71	vs	72 - 74

A Bonferroni analysis would carry out each comparison at the .05/4 = .0125 level versus the .05/10 = .005 level when all comparisons are done.

The following output was obtained from Minitab for doing these four comparisons, based on Wilcoxon-Mann-Whitney two sample tests (why?). Two year-groups are claimed to be different if the p-value is .0125 or below, or equivalently, if a 98.75% CI for the difference in population medians does not contain zero.

```
Mann-Whitney Confidence Interval and Test
Pre-63     N =  10     Median =       715.0
```

```
63-7       N =  13     Median =       780.0

Point estimate for ETA1-ETA2 is        15.0
98.8 Percent CI for ETA1-ETA2 is (-427.9,529.9)
W = 123.5

Test of ETA1 = ETA2  vs  ETA1 not = ETA2 is significant at 0.8524
The test is significant at 0.8524 (adjusted for ties)


Mann-Whitney Confidence Interval and Test

63-7       N =  13     Median =       780.0
68-9       N =  16     Median =       323.5

Point estimate for ETA1-ETA2 is       399.0
98.8 Percent CI for ETA1-ETA2 is (52.0,708.8)
W = 256.0

Test of ETA1 = ETA2  vs  ETA1 not = ETA2 is significant at 0.0080
The test is significant at 0.0080 (adjusted for ties)


Mann-Whitney Confidence Interval and Test

68-9       N =  16     Median =       323.5
70-1       N =  20     Median =       244.0

Point estimate for ETA1-ETA2 is        11.0
98.8 Percent CI for ETA1-ETA2 is (-171.0,206.0)

W = 300.0
Test of ETA1 = ETA2  vs  ETA1 not = ETA2 is significant at 0.9113
The test is significant at 0.9112 (adjusted for ties)


Mann-Whitney Confidence Interval and Test

70-1       N =  20     Median =       244.0
72-4       N =  19     Median =       160.0

Point estimate for ETA1-ETA2 is       130.0
98.8 Percent CI for ETA1-ETA2 is (6.1,285.9)

W = 497.5
Test of ETA1 = ETA2  vs  ETA1 not = ETA2 is significant at 0.0064
The test is significant at 0.0064 (adjusted for ties)
```

There are significant differences between the 1963-67 and 1968-69 samples, and between the 1970-71 and 1972-74 samples. You are 98.75% confident that the population median HC emissions for 1963-67 year cars is between 52 and 708.8 ppm greater than the population median for 1968-69 cars. Similarly, you are 98.75% confident that the population median HC emissions for 1970-71 year cars is between 6.1 and 285.9 ppm greater than the population median for 1972-74 cars.

It is not uncommon for researchers to combine data from groups not found to be significantly different. This is not, in general, a good practice. Just because you do not have sufficient evidence to show differences does not imply that you should treat the groups as if they are the same!

A Final ANOVA Comment

If the data distributions do not substantially deviate from normality, but the spreads are different across samples, you might consider the standard ANOVA followed with multiple comparisons using two-sample tests based on Satterthwaite's approximation.