Stat 345-002 Spring 2006 Key to Exam 2

1. (a) (4 pts)
   $P(X < 0) = P(\frac{X-15}{10} < \frac{0-15}{10}) = P(Z < -1.5) = 0.066807$

   (b) (4 pts)
   $P(X > 40) = P(\frac{X-15}{10} > \frac{40-15}{10}) = P(Z > 2.5) = 0.006210$

   (c) (4 pts)
   $P(X > x) = 0.90 \Leftrightarrow P(\frac{X-15}{10} > \frac{x-15}{10}) = 0.90 \Leftrightarrow P(Z > \frac{x-15}{10}) = 0.90$
   $\Leftrightarrow \frac{x-15}{10} = -1.28 \Leftrightarrow x = 15 - 1.28(10) = 2.2$

2. (a) (4 pts)
   You could just observe that this is a continuous uniform random variable over an interval of length 9 starting at 1, so $c = 10$. Formally, $\int_1^c \frac{1}{9}dx = 1 \Leftrightarrow \frac{x}{9}|_1^c = 1 \Leftrightarrow \frac{c-1}{9} = 1 \Leftrightarrow c = 10$

   (b) (4 pts)
   This is a continuous uniform random variable from 1 to 10 so $E(X) = \frac{1+10}{2} = 5.5$ and $V(X) = \frac{(10-1)^2}{12} = \frac{81}{12} = \frac{27}{4} = 6.75$. You can find these from the definition, but it is a lot of unnecessary work.

   (c) (4 pts)
   $P(X < 2) = \int_1^2 \frac{1}{9}dx = \frac{x}{9}|_x^2 = \frac{1}{9}$. (You could just observe that it is the area of a rectangle of dimensions 1 and 1/9).

3. (a) (6 pts)
   $P(X > 120) = P(\frac{X-np}{\sqrt{np(1-p)}} > \frac{120-np}{\sqrt{np(1-p)}}) \doteq P(Z > \frac{120-100}{\sqrt{75}}) = P(Z > 2.31) = 0.010444$

   (b) (6 pts)
   $P(X > 120) = P(\frac{X-\lambda}{\sqrt{\lambda}} > \frac{120-\lambda}{\sqrt{\lambda}}) \doteq P(Z > \frac{120-100}{10}) = P(Z > 2) = 0.022750$

4. (a) (4 pts)
   $P(2.5 < X < 3.5) = \int_{2.5}^{3.5} 0.5x - 1dx = \frac{x^2}{4} - x|_{2.5}^{3.5} = \frac{x(x-4)}{4}|_{2.5}^{3.5}$
   $= \frac{1}{4}(3.5(-.5) - 2.5(-1.5)) = \frac{1}{4}(3.75 - 1.75) = \frac{1}{2}$

   (b) (4 pts)
   $\mu = E(X) = \int_2^4 0.5x^2 - xdx = \frac{x^3}{6} - \frac{x^2}{2}|_2^4 = \frac{x^2(x-3)}{6}|_2^4 = \frac{20}{6} = \frac{10}{3}$

   $E(X^2) = \int_2^4 0.5x^3 - x^2dx = \frac{x^4}{8} - \frac{x^3}{3}|_2^4 = \frac{x^3(3x-8)}{24}|_2^4 = \frac{256+16}{24} = \frac{272}{24} = \frac{34}{3}$

   $\sigma^2 = V(X) = \frac{34}{3} - (\frac{10}{3})^2 = \frac{102-100}{9} = \frac{2}{9}$

   (c) (4 pts)
   We know $F(x) = 0,\ x < 2$ and $F(X) = 1,\ x \geq 4$.
   For $2 \leq x < 4,\ F(x) = \int_2^x 0.5w - 1dw = \frac{w^2}{4} - w|_2^x = \frac{x^2}{4} - x - 1 + 2 = \frac{x^2}{4} - x + 1$

   i.e. $F(x) = \begin{cases} 0 & x < 2 \\ \frac{x^2}{4} - x + 1 & 2 \leq x < 4 \\ 1 & 4 \leq x \end{cases}$. Many of you evaluated $\int_0^x 0.5w - 1dw$ for the middle term, thus claiming that most of the probabilities are negative.

5. (a) (6 pts)
   $P(T \geq 3) = \int_3^\infty \frac{1}{6}e^{-t/6}dt = -e^{-t/6}|_3^\infty = 0 + e^{-1/2} = 0.6065$

(b) (6 pts)
You have X is the number of Successes (Success = computer fails) in 10 independent trials, each with probability of Success $p = P(T \leq 3) = 0.3935$. Therefore $X \sim \text{binomial}(n = 10, p = 0.3935)$ and you need to find $P(X \geq 2) =$
$1 - P(X < 2) = 1 - p(0) - p(1) = 1 - (0.6065)^{10} - 10(0.3935)(0.6065)^9 = 0.9496$

6. (a) (6 pts)
If X = No. Flaws in 50 panels then $X \sim \text{Poisson}(\lambda = 50(.02) = 1)$
$P(X = 0) = \frac{e^{-1}1^0}{0!} = e^{-1} = 0.3679$

(b) (7 pts)
The chance a single panel has one or more flaws is 1 - probability panel has no flaws $= 1 - \frac{e^{-0.02}(0.02)^0}{0!} = 1 - e^{-0.02} = 0.0198$. X = No. of panels to inspect until you see one that is defective, so $X \sim \text{geometric}(p = 0.0198)$ You actually need to worry about the panels being independent, but the Poisson process will give that to you (you should worry, but it's ok). $E(X) = \frac{1}{p} = \frac{1}{0.0198} = 50.5$

If you used the Poisson mean as p then it was a lucky (close) guess. Suppose the Poisson mean was 1.5 – would that still work as a probability?

7. (a) (7 pts)
Fixed number (50) of panels. Again, you need to assume independence. X = No. of panels with flaws is binomial n = 50 and p = 0.0198.
$P(X \leq 2) = p(0) + p(1) + p(2)$
$= (0.9802)^{50} + 50(0.9802)^{49}(0.0198) + \frac{50 \cdot 49}{2}(0.9802)^{48}(0.0198)^2 = 0.92338$
If you used the Poisson mean as the probability it was a lucky guess again.

(b) (7 pts)
I wrote on the board and announced during the exam that this was *without* replacement. If X is the number of panels in the sample with flaws then X is hypergeometric with $N = 50,\ n = 10,\ K = 5$ so
$P(X = x) = \frac{\binom{5}{x}\binom{45}{10-x}}{\binom{50}{10}};\ x = 0, 1, 2, 3, 4, 5.$ Want $P(X > 0) = 1 - P(X = 0)$
$= 1 - \frac{\binom{45}{10}}{\binom{50}{10}} = 1 - \frac{45 \cdot 44 \cdot 43 \cdots 37 \cdot 36}{50 \cdot 49 \cdot 48 \cdots 42 \cdot 41} = 1 - \frac{40 \cdot 39 \cdot 38 \cdot 37 \cdot 36}{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46} = 1 - 0.31056 = 0.68944$

8. (a) (6 pts)
If $X = \#$ of assemblies to check to get five defectives, then $X \sim$ negative binomial with $p = 0.01$ and $r = 5$. Thus $E(X) = \frac{5}{0.01} = 500$.

(b) (7 pts)
If you check n assemblies the probability of at least one defective is 1 - probability of no defectives $= 1 - (0.99)^n$. You want n large enough that $1 - (0.99)^n > 0.95 \Leftrightarrow (0.99)^n < 0.05 \Leftrightarrow n \log 0.99 < \log 0.05 \Leftrightarrow n > \frac{\log 0.05}{\log 0.99}$ (note reverse inequality since $\log 0.99 < 0$) $\Leftrightarrow n > 298.07 \Leftrightarrow n \geq 299$.