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Effect of deviatoric nonassociativity on the failure prediction for elastic–plastic materials

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ABSTRACT

This paper presents a theoretical study of the effect of nonassociativity of the plastic flow rule on the critical plastic modulus for discontinuous bifurcation in an elastic–plastic material. Nonassociativity in both the spherical and the deviatoric spaces are considered, with an emphasis on the effect of nonassociativity in the deviatoric space. A particular form of nonassociativity in the deviatoric space is introduced, where the projections of the plastic flow direction and the normal to the yield surface are assumed to have the same length but the projection of plastic flow direction is allowed to lag that of the normal by an angle. It is shown that even for the simple yield surface of von Mises, nonassociativity in the deviatoric space can lead to a bifurcation for a load parameter significantly lower than the value predicted with an associated flow rule.

1. Introduction

Due to its close connection with the failure prediction of inelastic materials and structures, the study of strain location and instability of elastic–plastic materials has remained as an active research topic in recent years (Rudnicki and Rice, 1975; Rice, 1976; Needleman and Tvergaard, 1982; Ottosen and Runesson, 1991a,b,c; Larsson et al., 1993; Neilsen and Schreyer, 1993; Schreyer and Neilsen, 1996a,b; Szabo, 2000; Schreyer, 2007). Researches on material instability analysis date back to the classical work of Hadamard (1903) on the stability of elastic motion. Hill (1958, 1962), Mandel (1966), Rudnicki and Rice (1975), Rice (1976) followed that line and developed a criterion for localization in elastic–plastic solids by assuming that the deformation localizes into a planar band and that the jump in strain rate field across the band satisfies the Maxwell kinematic compatibility conditions. The criterion states that localization occurs when the lowest eigenvalue of the acoustic tensor, which depends on both the direction and the tangent modulus of the material, reduces to zero (Rudnicki and Rice, 1975; Rice, 1976). For the static problem, the existence of a zero eigenvalue of the acoustic tensor corresponds to the loss of ellipticity of the incremental governing differential equation, which in turn causes nonuniqueness of the solution of the boundary value problem (Kreiss and Lorenz, 1989; Belytschko et al., 2000). For the dynamic problem, on the other hand, the existence of a zero eigenvalue of the acoustic tensor corresponds to a zero plastic wave speed, which implies that any perturbation to the problem will not vanish after any period of time. In the analysis, it has been assumed that the materials on both sides of the band have the same constitutive relation. The essential idea behind all this work is that, for a static equilibrium or dynamic wave propagation problem of elastic–plastic material, if one of the wave speeds of a dynamic perturbation is imaginary, then the problem is ill-posed, or unstable in the sense of Hadamard.

Effects of various features in the constitutive description of the materials (vertices in the yield surface, nonassociativity in the spherical space of the plastic flow rule) on localization have also been studied (Rudnicki and Rice, 1975; Rice, 1976; Pan and Rice, 1983; Needleman and Tvergaard, 1982). Those pioneering researches show that the presence of the features in the constitutive model, which can be regarded as deviations from a smooth yield surface with associated flow rule, promotes the initiation of localization, giving better comparisons with experimental data. Furthermore, features such as the development of vertices in the yield surface and nonassociativity in the spherical space are consistent with the micromechanics of the material behavior. For example, as discussed by Rice (1976) and by Pan and Rice (1983), for crystalline materials vertices in the yield surface are a natural consequence of the activation of multiple slip systems in a rate-independent crystal, and for geological materials nonassociativity in the spherical space of the flow rule is the result of the Coulomb frictional nature of the yielding of a material.

In the current work, we study the effect of nonassociativity in the deviatoric space on the initiation and orientation of strain localization. Deviatoric nonassociativity has been considered by

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several researchers in recent years (Loret, 1992; Ottosen and Runesson, 1991a,b,c; Szabo, 2000; Kobayashi, 1998, 2003, 2010). In particular, Loret (1992) has studied the effect of deviatoric nonassociativity on the possibility of flutter instability in geomaterials and stated the validity of deviatoric associativity for such materials is "an open problem". For ductile materials, Kobayashi and coworkers (1998, 2003, 2010), in a study of localization of polycrystalline A1070 aluminum, have used a plastic flow rule that is nonassociated in the deviatoric plane. Kobayashi (2010) attributes this nonassociativity to the elastoplastic coupling between the plastic deformation and degradation of elastic stiffness of materials. Desrues and Chambon (1989) have proposed a formulation in which the plastic strain rate is given as a linear combination of the current stress and the current stress rate, which can be interpreted as deviatoric nonassociativity if a smooth yield surface is used. In addition to providing a different stress–strain response, nonassociativity can have significant implications concerning the orientation and point of initiation of a discontinuous bifurcation. The analysis in this paper provides a detailed analysis for nonassociativity in the deviatoric space based on von Mises plasticity. Surprisingly, perhaps, the degree of nonassociativity has little effect on the orientation of the plane of localization but a very significant effect on the critical load level at which bifurcation initiates. Analogous to the results of Rudnicki and Rice (1975) for nonassociativity with respect to the spherical part of the stress tensor, current predictions show that nonassociativity in the deviatoric space can move the initiation of the bifurcation from the softening region well back into the hardening regime.

The paper proceeds as follows. After a brief review of the key concepts and definitions needed for a discussion of instability and ill-posedness associated with problems of inelastic materials, Section 2 summaries the key results of a discontinuous bifurcation analysis of elastic–plastic materials with nonassociated flow. Section 3 introduces a special form of nonassociated plastic flow, where it is assumed that the projections of the plastic strain rate direction and the normal to the yield surface have the same length in the deviatoric plane. With this assumption, the nonassociativity in the deviatoric plane is completely characterized by the angle between the projections. The effect of the degree of nonassociativity on the failure predictions (both the failure orientation and critical plastic modulus) is studied in Section 4, where an example of the von Mises yield surface for various loading paths is provided. The result in this section indicates that nonassociativity in the deviatoric plane has little effect on the orientation of the failure plane but a large reduction in the stress at which bifurcation occurs is possible. The paper ends with a summary and some concluding remarks given in Section 5.

2. Summary of discontinuous bifurcation analysis

2.1. Initial comments and notation

Here we provide a synopsis of the theory to form the basis of subsequent developments. The approach taken in this work to address the bifurcation and instability problem of an inelastic material is the dynamic perturbation analysis, which has been shown by Rice (1976) to be equivalent to the classic bifurcation analysis of Rudnicki and Rice (1975) based on the consideration of traction continuity and Maxwell compatibility requirement. The use of dynamic perturbation allows us to address the bifurcation and stability problem of materials by applying the techniques and results in the mathematical literature on the well-posedness of initial-boundary value problems. The same dynamic perturbation approach has been successfully applied to the buckling and instability analysis of elastic structures.

For the sake of compactness, the following direct notation for vector and tensor operations (e.g. Gurtin, 1981; Schreyer and Neilsen, 1996a,b; Belytschko et al., 2000) will be used in most of the paper:

$$
\begin{aligned}
&\mathbf{i} \equiv \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \qquad \mathbf{I} \equiv \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l,\\
&\mathbf{u} \otimes \mathbf{v} \equiv u_i v_j \mathbf{e}_i \otimes \mathbf{e}_j, \qquad \mathbf{A} \otimes \mathbf{B} \equiv A_{ij}B_{kl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l,\\
&\mathbf{u} \cdot \mathbf{v} \equiv u_k v_k, \qquad \mathbf{A} \cdot \mathbf{u} \equiv A_{ik}u_k\mathbf{e}_i, \qquad \mathbf{u} \cdot \mathbf{A} = \mathbf{A}^T \cdot \mathbf{u} \equiv A_{ki}u_k\mathbf{e}_i,\\
&\mathbf{A} \cdot \mathbf{B} \equiv A_{ik}B_{kj}\mathbf{e}_i \otimes \mathbf{e}_j, \qquad \mathbf{T} : \boldsymbol{\varepsilon} \equiv T_{ijkl}\varepsilon_{kl}\mathbf{e}_i \otimes \mathbf{e}_j,\\
&tr\,\mathbf{A} = \mathbf{i} : \mathbf{A} = A_{ii}, \qquad \mathbf{A} : \mathbf{B} \equiv tr(\mathbf{A} \cdot \mathbf{B}^T) = A_{ik}B_{ik},
\end{aligned}
$$

where $\mathbf{i}$ is the second-order identity tensor; $\mathbf{I}$, the fourth-order identity tensor; δ_{ij}, the Kronecker delta; $\mathbf{e}_i$, an arbitrary orthonormal basis; $\mathbf{u}$, $\mathbf{v}$, vectors; $\mathbf{A}$, $\mathbf{B}$, symmetric, second-order tensors; and $\mathbf{T}$, a fourth-order tensor.

2.2. Dynamic perturbation

Consider the stability of a homogenous solution to an initial-boundary value problem involving a rate-independent, elastic–plastic material. Let a dynamic perturbation, which is small compared to the homogenous solution itself, be imposed on the solution. Then the stability of the solution can be determined by the behavior of the perturbation with time: the solution is structural unstable if the perturbation can grow with time (e.g., Troger and Steindl, 1991; Seydel, 1994). Worse yet, if the perturbation can approach infinity in a finite time, then the problem is said to be ill-posed (Kreiss and Lorenz, 1989). This is the classical Hadamard instability. In this paper, we refer to structural instability as just instability and the Hadamard instability as ill-posedness.

For the study of stability it is sufficient to consider only the perturbation with the following planar wave form (Rice, 1976):

$$\mathbf{v}(\mathbf{r}, t) = Re\{\mathbf{v}_0 e^{ik(\mathbf{n}\cdot\mathbf{r} - ct)}\}, \tag{1}$$

where $\mathbf{v}$ is the perturbation to the particle velocity, $\mathbf{r}$ the position vector, t time, $\mathbf{v}_0$ the magnitude (vector) of the initial perturbation (also the wave polarization directions, Auld, 1990), k wave number ($\lambda = 2\pi/k$, the wave length), $\mathbf{n}$ unit normal to the planar wave front ($\mathbf{n} \cdot \mathbf{n} = 1$), and c the wave speed. It is seen from Eq. (1) that the initial perturbation has been assumed to be

$$\mathbf{v}(\mathbf{r}, 0) = Re\{\mathbf{v}_0 e^{ik\mathbf{n}\cdot\mathbf{r}}\} = \mathbf{v}_0 \cos kx, \tag{2}$$

where $x \equiv \mathbf{n} \cdot \mathbf{r}$ is the coordinate along the normal to the planar wave front. Since a perturbation can have an arbitrary wave length and can be along any direction, we need to consider all values of wave number k and all wave normal ($\mathbf{n}$). In a numerical analysis using finite elements, $\mathbf{v}(\mathbf{r}, 0) = \mathbf{v}_0 \cos kx$ can be thought as the error introduced by discretization. Mesh refinement then corresponds to an increase in the wave number k (or decrease in the wave length).

It follows from Eq. (1) that the solution is stable if and only if the wave speed c remains real for all wave numbers k and all directions $\mathbf{n}$. The wave speed c is given by the following eigenvalue problem (Rice, 1976; Ottosen and Runesson, 1991a,b,c; Larsson et al., 1993; Schreyer and Neilsen, 1996a,b; Belytschko et al., 2000)

$$\mathbf{A}(\mathbf{n}) \cdot \mathbf{v}_0 = \rho c^2 \mathbf{v}_0, \tag{3}$$

where $\mathbf{A}(\mathbf{n})$ is the acoustic tensor corresponding to the direction $\mathbf{n}$:

$$\mathbf{A}(\mathbf{n}) \equiv \mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n}, \tag{4}$$

and $\mathbf{T}$ is the tangent modulus (tensor) of the material. For a rate-independent elastic–plastic solid, when the corotational terms in the objective stress rate are neglected (Rice, 1976; Loret, 1992), the Cauchy stress rate $\dot{\boldsymbol{\sigma}}$ is related to the total strain rate (the symmetric part of the velocity gradient) by (e.g, Ottosen and Runesson,

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1991b,c; Loret, 1992; Schreyer and Neilsen, 1996a,b; Zyczkowski, 1998; Szabo, 2000)

$$\dot{\boldsymbol{\sigma}} = \mathbf{T} : \dot{\boldsymbol{\varepsilon}} = \mathbf{T} : (\mathbf{v}\nabla). \tag{5}$$

According to Loret (1992), in many practical problems, especially in civil engineering and geomechanical analyses, neglecting the corotational terms in the objective stress rate is not a severe restriction. Since the intended applications of the current analysis are for geomaterials, we will not consider the effects of the corotational terms in this work. For problems where the corotational terms become important, the ordinary time rate of stress in the Eq. (5) should be replaced by an objective stress rate (such as the Jaumann rate, e.g., Rice, 1976).

It follows from Eqs. (1) and (3) that the wave speeds c and the polarization directions $\mathbf{v}_0$ are related to the eigenvalues and eigenvectors of the acoustic tensor $\mathbf{A}(\mathbf{n})$. For an elastic–plastic material, $\mathbf{A}(\mathbf{n})$ is a function of both the propagation direction and the plastic deformation. It follows from Eq. (3) that when the acoustic tensor becomes singular [$\det \mathbf{A}(\mathbf{n}) = 0$] for some critical direction $\mathbf{n}^c$, the lowest wave speed c reduces to zero and perturbations along that direction would not propagate, but are trapped in the material causing instability. The condition for which $\mathbf{A}(\mathbf{n})$ is singular is called the loss of ellipticity condition. Based on the Maxwell compatibility condition for a jump in the strain field, Rudnicki and Rice (1975), Rice (1976) found that a discontinuous bifurcation in the deformation of an elastic–plastic material occurs when the loss of ellipticity condition is met. It was shown by Rice (1976) that the loss of ellipticity condition also corresponds to the material instability.

Upon further plastic deformation beyond the loss of ellipticity condition, if the lowest eigenvalue of $\mathbf{A}(\mathbf{n}^c)$ becomes negative ($\lambda_3 < 0$), the wave speed becomes imaginary: $c = \pm i\sqrt{|\lambda_3|/\rho}$. The perturbation then becomes

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{v}_0 Re\{e^{ikx}e^{-ikct}\} = \mathbf{v}_0 \cos kx e^{\pm k\sqrt{|\lambda_3|/\rho}t}, \tag{6}$$

that is, the perturbation with a finite wave number k grows exponentially with time (this corresponds to the structural instability defined earlier); furthermore, even within a finite time $t = t_0 < \infty$, the perturbation grows unbounded as the wave number k increases. As mentioned earlier, mesh refinement in a finite element analysis introduces perturbations with increasing wave number k (or decreasing wave length). Therefore, when the lowest eigenvalue of $\mathbf{A}(\mathbf{n}^c)$ becomes negative, the numerical solution starts to lose convergence as the mesh is refined even for a finite time, as has been observed in numerical simulations (e.g. Sluys, 1992). This is the Hadamard instability and the problem is said to be ill-posed.

The loss of ellipticity condition, $\det \mathbf{A}(\mathbf{n}) = 0$, is used in the paper as the criterion for discontinuous bifurcation and initiation of material failure. We now derive the expressions for the tangent modulus $\mathbf{T}$ and the acoustic tensor $\mathbf{A}(\mathbf{n})$.

2.3. Tangent modulus

Consider the plastic deformation of the material. The stress rate can be written as (e.g., Ottosen and Runesson, 1991a,b,c; Loret, 1992; Schreyer and Neilsen, 1996a,b; Szabo, 2000)

$$\dot{\boldsymbol{\sigma}} = \mathbf{E} : \dot{\boldsymbol{\varepsilon}}^{\mathbf{e}} = \mathbf{E} : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p), \tag{7}$$

where $\dot{\boldsymbol{\varepsilon}}^{\mathbf{e}}$ and $\dot{\boldsymbol{\varepsilon}}^{\mathbf{p}}$ are, respectively, the elastic strain rate and plastic strain rate, and $\mathbf{E}$ is the elasticity tensor, which is anisotropic in general. Without loss of generality, the elasticity tensor $\mathbf{E}$ is assumed to possess both minor and major symmetries ($E_{ijkl} = E_{jikl} = E_{jilk} = E_{klij}$; recall that $\mathbf{E}$ can be assumed to possess major symmetries when the elastic strain energy density function is defined for the material).

It is assumed that for a rate-independent elastic–plastic material, there exists a yield surface $f(\boldsymbol{\sigma}, \mathbf{q}) = 0$, inside of which the plastic strain rate is zero ($\dot{\boldsymbol{\varepsilon}}^p = 0$). The vector $\mathbf{q}$ contains hardening variables. The evolution equations for the plastic deformation and hardening are given by

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\mathbf{M}(\boldsymbol{\sigma}, \mathbf{q}), \tag{8}$$

$$\dot{\mathbf{q}} = -\dot{\lambda}\mathbf{h}(\boldsymbol{\sigma}, \mathbf{q}), \tag{9}$$

where $\dot{\lambda}$ is a plasticity parameter determined by the consistency condition $\dot{f}(\boldsymbol{\sigma}, \mathbf{q}) = 0$, which requires that the stress state remain on the yield surface for plastic loading ($f(\boldsymbol{\sigma}, \mathbf{q}) = 0$ and $\partial_\sigma f : \mathbf{E} : \dot{\boldsymbol{\varepsilon}} = \partial f(\boldsymbol{\sigma}, \mathbf{q})/\partial\boldsymbol{\sigma} : \mathbf{E} : \dot{\boldsymbol{\varepsilon}} > 0$). A specific plasticity model is defined when the expressions $f(\boldsymbol{\sigma}, \mathbf{q})$, $\mathbf{M}(\boldsymbol{\sigma}, \mathbf{q})$, and $\mathbf{h}(\boldsymbol{\sigma}, \mathbf{q})$ are provided for the yield surface, the plastic flow rule, and the hardening function, respectively. The tangent modulus (or tensor) for the material defined in Eq. (5) can be written as (e.g., Simo and Hughes, 1998)

$$\mathbf{T} = \mathbf{E} - \mathbf{E}^p = \mathbf{E} - \psi(\mathbf{E} : \mathbf{M}) \otimes (\mathbf{N} : \mathbf{E}), \tag{10}$$

where $\mathbf{N}$ is the normal to the yield surface

$$\mathbf{N}(\boldsymbol{\sigma}, \mathbf{q}) \equiv \partial_\sigma f = \frac{\partial f(\boldsymbol{\sigma}, \mathbf{q})}{\partial \boldsymbol{\sigma}}. \tag{11}$$

In Eq. (10) the scalar ψ is defined by

$$\psi \equiv \frac{1}{\mathbf{N} : \mathbf{E} : \mathbf{M} + E_p} = \frac{1}{K + E_p}, \quad K \equiv \mathbf{N} : \mathbf{E} : \mathbf{M}, \tag{12}$$

where E_p is the plastic (hardening) modulus

$$E_p \equiv \partial_{\mathbf{q}} f \cdot \mathbf{h} = \frac{\partial_q f(\boldsymbol{\sigma}, \mathbf{q})}{\partial \mathbf{q}} \cdot \mathbf{h}. \tag{13}$$

It follows from Eq. (10) that if the flow rule is associative, $\mathbf{M}(\boldsymbol{\sigma}, \mathbf{q}) = \mathbf{N}(\boldsymbol{\sigma}, \mathbf{q})$, that is, if the plastic strain rate is along the normal to the yield surface, then the tangent tensor $\mathbf{T}$ also has major symmetry ($T_{ijkl} = T_{klij}$). However, for a nonassociative flow rule, $\mathbf{M}(\boldsymbol{\sigma}, \mathbf{q}) \neq \mathbf{N}(\boldsymbol{\sigma}, \mathbf{q})$; consequently, the tangent tensor $\mathbf{T}$ does not have major symmetry even though the elasticity tensor $\mathbf{E}$ possesses both minor and major symmetries. It is noted that the formulations presented so far are rather general, allowing for anisotropies in both elasticity and plasticity and for the flow rule to be nonassociative. Isotropic plasticity is implied when the yield function $f(\boldsymbol{\sigma}, \mathbf{q})$ is given in terms of stress invariants. Conversely, if the yield function cannot be expressed in terms of stress invariants only, then the plasticity part of the model is anisotropic.

2.4. Acoustic tensor

Substitution of the tangent tensor in Eq. (10) into Eq. (4) yields the expression for the acoustic tensor of the elastic–plastic material

$$\mathbf{A}(\mathbf{n}) = \mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n} = \mathbf{A}_E - \mathbf{A}_P, \tag{14}$$

where $\mathbf{A}_E$ and $\mathbf{A}_P$ are respectively the acoustic tensors associated with elastic and plasticity terms (e.g., Larsson et al., 1993)

$$\mathbf{A}_E \equiv \mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n}, \tag{15a}$$

$$\mathbf{A}_P \equiv \mathbf{n} \cdot \mathbf{E}^p \cdot \mathbf{n} = \psi\mathbf{b} \otimes \mathbf{a}, \tag{15b}$$

where

$$\mathbf{b} \equiv \mathbf{n} \cdot (\mathbf{E} : \mathbf{M}), \qquad (b_i = n_j E_{ijkl} M_{kl}), \tag{16a}$$

$$\mathbf{a} \equiv (\mathbf{N} : \mathbf{E}) \cdot \mathbf{n}, \qquad (a_j = N_{mn} E_{mnjl} n_l = n_l E_{jlmn} N_{mn}). \tag{16b}$$

Since the elasticity tensor $\mathbf{E}$ is assumed to have major symmetry, $\mathbf{A}_E$ is always symmetric. However, if the plastic flow rule is nonassociative, then $\mathbf{A}_P$ and hence $\mathbf{A}$ are not symmetric. For an associated plastic flow rule, $\mathbf{M}(\boldsymbol{\sigma}, \mathbf{q}) = \mathbf{N}(\boldsymbol{\sigma}, \mathbf{q})$; consequently, $\mathbf{b} = \mathbf{a}$, and the

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acoustic tensor $\mathbf{A}$ becomes $\mathbf{A}(\mathbf{n}) = \mathbf{n}\cdot\mathbf{En} - \psi\mathbf{a}\otimes\mathbf{a}$, which agrees with the previous work on associated plasticity (e.g., Schreyer and Neilsen, 1996a,b). On the other hand, if the plastic flow rule is nonassociative, $\mathbf{M}(\boldsymbol{\sigma},\mathbf{q})\neq\mathbf{N}(\boldsymbol{\sigma},\mathbf{q})$, then $\mathbf{b}\neq\mathbf{a}$ and the acoustic tensor $\mathbf{A}$ is not symmetric. The focus of this paper is on the materials for which the plastic flow rule is nonassociative.

2.5. Plastic wave speeds

In this paper, we only consider elastic–plastic materials with isotropic elasticity. For an isotropic material the elasticity tensor defined in Eq. (7) can be written as (e.g., Schreyer and Zuo, 1995)

$$\mathbf{E} = \bar{\lambda}\mathbf{i}\otimes\mathbf{i} + 2G\mathbf{I}, \qquad (17)$$

where $\bar{\lambda}$ and G are the Lame constants of the materials; $\mathbf{i}$ and $\mathbf{I}$ are, respectively, the second- and fourth-order identity tensors defined previously. With the limit of an isotropic elasticity, the eigenvalues of the acoustic tensor defined in Eq. (14) can be solved analytically (e.g., Ottosen and Runesson, 1991a,b,c; Brannon and Drugan, 1993; Zuo, 1995). For completeness, the key results of these studies are briefly summarized below.

Along an arbitrary propagation direction $\mathbf{n}$, the eigenvalue of $\mathbf{A}$ are (the details are given in, e.g., Ottosen and Runesson, 1991c):

$$\delta_3 = G, \qquad (18a)$$

$$\delta_{1,2} = \frac{1}{2}\left[\bar{\lambda} + 3G - \psi(\mathbf{a}\cdot\mathbf{b}) \pm \sqrt{\left[\bar{\lambda} + G + \psi(\mathbf{a}\cdot\mathbf{b})\right]^2 - 4(\bar{\lambda}+G)\psi(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{n})}\right]. \qquad (18b)$$

The corresponding plastic wave speeds are $\sqrt{G/\rho}$, $\sqrt{\delta_1/\rho}$, and $\sqrt{\delta_2/\rho}$.

For applications where the elastic anisotropy cannot be neglected, simple closed-form expressions for the eigenvalues of $\mathbf{A}$, similar to those given here, are not available for an arbitrary propagation direction. However, it is still possible to make some estimates on the eigenvalues, as reported in a recent work (Zuo, 2010).

2.6. Loss of ellipticity

The determinant of a tensor is just the products of its eigenvalues. Therefore

$$Det(\mathbf{A}) = G\delta_1\delta_2 = G\left[G(\bar{\lambda}+2G) - m\psi\right], \qquad (19)$$

where

$$m(\mathbf{n}) \equiv (\bar{\lambda}+2G)(\mathbf{a}\cdot\mathbf{b}) - (\bar{\lambda}+G)(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{n}). \qquad (20)$$

Recall that loss of ellipticity occurs when $Det(\mathbf{A}) = 0$. As stated earlier, a discontinuous bifurcation occurs when the ellipticity condition is lost. It follows that the loss of ellipticity occurs when the parameter ψ reaches a critical value

$$\psi^* = \frac{G(\bar{\lambda}+2G)}{m(\mathbf{n})}. \qquad (21)$$

Recall from Eq. (12) that $\psi \equiv \frac{1}{\mathbf{N}:\mathbf{E}:\mathbf{M}+E_p} = \frac{1}{K+E_p}$. Therefore a bifurcation occurs when the plastic modulus assumes the value

$$E_p^*(\mathbf{n}) = \frac{m(\mathbf{n})}{G(\bar{\lambda}+2G)} - K. \qquad (22)$$

An equivalent form of Eq. (22) was derived by Larsson et al. (1993) by taking an alternative approach. For a given constitutive model, $\mathbf{E}$ (hence $\bar{\lambda}$ and G), $\mathbf{M}$, and $\mathbf{N}$ are known, so $\mathbf{a}$ and $\mathbf{b}$ are functions of the direction $\mathbf{n}$. It follows from Eq. (22) that the critical modulus E_p^* at which bifurcation occurs also depends solely on $\mathbf{n}$. For a given stress state, there will be a critical orientation $\mathbf{n}^c$ which yields the highest value of E_p^* (or the lowest value of a loading parameter). This critical orientation $\mathbf{n}^c$ corresponds to the normal to the plane of localization and is defined here as the failure direction (normal to the failure plane); the corresponding plastic modulus is called the critical modulus.

For a nonassociated flow rule, since the acoustic tensor $\mathbf{A}$ is not symmetric, another possible failure mode of the material is flutter instability as indicated by Rice (1976), in addition to the discontinuous bifurcation discussed above. By flutter instability, it is meant that the acoustic tensor admits a pair of complex eigenvalues without going through the origin. Though flutter instability of flexible structures under nonconservative loading, for example, bending flutter of slender missile under an end rocket thrust (Langthjem and Sugiyama, 1999), has been a major topic in aeroelasticity (Fung, 1955), flutter instability in elastic–plastic materials has only received limited amount of attention (An and Schaeffer, 1992; Loret, 1992; Bigoni and Willis, 1994; Bigoni, 1995). The possibility of flutter instability is not considered in this paper.

2.7. Closing comments

Ottosen and Runesson (1991c) and Zuo (1995) have found the analytical expressions for the critical orientation $\mathbf{n}^c$ under a rather general form of nonassociativity (it is only required that $\mathbf{M}$ and $\mathbf{N}$ share the same principal directions which are satisfied by almost all practical constitutive models and which is assumed in the remainder of the paper). For a specific constitutive equation, the yield surface $f(\boldsymbol{\sigma},\mathbf{q}) = 0$ and plastic flow rule are known. The normal to the yield surface, $\mathbf{N}$, and the direction of the plastic strain rate, $\mathbf{M}$, can be found for given stress state $\boldsymbol{\sigma}$ and the hardening function $\mathbf{q}$. Then the critical orientation $\mathbf{n}^c$ and the failure direction and the critical modulus can be calculated using the formulas in Ottosen and Runesson (1991c) and Zuo (1995). The formulas given in Zuo (1995) are used to generate the numerical results presented in Section 4.

In their general formulations for the plastic wave speeds and for the discontinuous bifurcation analysis, Ottosen and Runesson (1991a,b,c) have considered nonassociativity in both the spherical and deviatoric spaces. However, in the material models used for their numerical examples they have limited nonassociativity to the spherical space only. In the following (Sections 3 and 4), we will focus on nonassociativity in the deviatoric space and provide a detailed analysis of simple von Mises model with deviatoric nonassociativity.

There are materials for which the yield surfaces are available, but the plastic flow direction is not well defined. In what follows, we assume that $\mathbf{N}$ is known, and $\mathbf{M}$ is related to $\mathbf{N}$ through some parameters defined to represent various degrees of nonassociativity.

3. Representation of the nonassociativity

3.1. Initial comments

In order to present the final results in as simple a manner as possible, we introduce coordinates in the deviatoric plane. We first consider general nonassociativity where the plastic flow rule is allowed to be nonassociative in both the spherical and the deviatoric spaces. We then focus on a restricted case of nonassociativity in which the plastic flow direction is assumed to lag the normal of the yield surface by an angle in the deviatoric plane. With this assumption, the effect of nonassociativity in the deviatoric plane can be conveniently presented in terms of a single angle parameter.

3.2. General nonassociativity

Suppose the plastic flow rule is nonassociative ($\mathbf{M}\neq\mathbf{N}$). Let the plastic flow direction $\mathbf{M}$ and the normal to the yield surface $\mathbf{N}$ be decomposed into the spherical and deviatoric parts

$$\mathbf{M} = \frac{1}{3}(tr\,\mathbf{M})\mathbf{i} + \mathbf{M}^d, \tag{23a}$$

$$\mathbf{N} = \frac{1}{3}(tr\,\mathbf{N})\mathbf{i} + \mathbf{N}^d, \tag{23b}$$

where $\mathbf{M}^d$ and $\mathbf{N}^d$ are the deviatoric parts of $\mathbf{M}$ and $\mathbf{N}$, respectively. It is usually assumed in the literature on nonassociated plasticity that $tr\,\mathbf{M} \neq tr\,\mathbf{N}$ but $\mathbf{M}^d = \mathbf{N}^d$. In other words, the nonassociativity is only allowed in the spherical space, but not in the deviatoric one. Here we extend the nonassociativity into the deviatoric space as well. Since the spherical part a tensor is completely determined by its trace, only a scalar is needed to characterize the nonassociativity in the spherical space

$$tr\,\mathbf{M} = n\,tr\,\mathbf{N}. \tag{24}$$

In general, five parameters are needed to represent a deviatoric part of a tensor. The concept of the Pi (or deviatoric) plane for stress tensor was introduced and has been conveniently used in the study of yield surfaces (Hill, 1950). The Pi plane is defined as the plane that forms equal angles with the three principal directions of the stress tensor.

Since we have assumed that $\mathbf{M}$ and $\mathbf{N}$ share the same principal directions, it follows that the Pi planes for $\mathbf{M}$ and $\mathbf{N}$ are the same (Fig. 1a). Let the principal directions of $\mathbf{M}$ and $\mathbf{N}$ be $\mathbf{e}_i$ $(i = 1, 2, 3)$. Since the projections of $\mathbf{e}_i$ on the Pi plane, $\bar{\mathbf{e}}_i$ $(i = 1, 2, 3)$, form angles of 120 degrees with each other, as shown in Fig. 1b, it is more convenient to introduce an orthonormal base, $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$, with $\mathbf{q}_3$ normal to the Pi plane and $\mathbf{q}_2$ in the $\bar{\mathbf{e}}_2$ direction, as shown in Fig. 1b (the details of the expressions for the $\mathbf{q}_i$ are given in Zuo, 1995).

The coordinates of $\mathbf{N}$ in its Pi plane, $\bar{n}_1$ and $\bar{n}_2$, are related to its principal values, N_1, N_2, and N_3 through the following equations (Hill, 1950):

$$\bar{n}_1 = \frac{1}{\sqrt{2}}(N_1 - N_3), \tag{25a}$$

$$\bar{n}_2 = \frac{1}{\sqrt{6}}(2N_2 - N_1 - N_3). \tag{25b}$$

Alternatively, the projection of $\mathbf{N}$ in the Pi plane can be represented by its length in the plane and the angle that the projection makes with the horizontal axis (the Lode angle, Hill, 1950):

$$\bar{n} = \sqrt{\bar{n}_1^2 + \bar{n}_2^2} = \sqrt{\mathbf{N}^d : \mathbf{N}^d} = \sqrt{\frac{2}{3}\left(N_1^2 + N_2^2 + N_3^2 - N_1N_2 - N_2N_3 - N_3N_1\right)}, \tag{26a}$$

$$\theta = \tan^{-1}\frac{\bar{n}_2}{\bar{n}_1} = \tan^{-1}\frac{(2N_2 - N_1 - N_3)}{\sqrt{3}(N_1 - N_3)}. \tag{26b}$$

3.3. Restricted form of deviatoric nonassociativity

For the sake of simplicity, it is further assumed that the length of the projection of $\mathbf{M}$ is the same as that of $\mathbf{N}$ in the Pi plane, that is,

$$\bar{m} = \bar{n}. \tag{27}$$

Now the only parameter left to characterize the deviatoric nonassociativity is the angle between $\mathbf{M}$ and $\mathbf{N}$ in the Pi plane. Let α denote the angle by which $\mathbf{M}$ lags $\mathbf{N}$ (Fig. 1b and Fig. 2). Then, the following expression for the eigenvalues of $\mathbf{M}$ can be found in terms of those of $\mathbf{N}$ through two parameters n and α, which, respectively, characterize the degree of nonassociativity in the spherical and the deviatoric spaces (the details are given in Zuo, 1995):

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \frac{1}{3}\begin{Bmatrix} 2\cos\alpha N_1 - \left(\cos\alpha - \sqrt{3}\sin\alpha\right)N_2 - \left(\cos\alpha + \sqrt{3}\sin\alpha\right)N_3 \\ 2\cos\alpha N_2 - \left(\cos\alpha - \sqrt{3}\sin\alpha\right)N_3 - \left(\cos\alpha + \sqrt{3}\sin\alpha\right)N_1 \\ 2\cos\alpha N_3 - \left(\cos\alpha - \sqrt{3}\sin\alpha\right)N_1 - \left(\cos\alpha + \sqrt{3}\sin\alpha\right)N_2 \end{Bmatrix} + n\frac{N_1 + N_2 + N_3}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}. \tag{28}$$

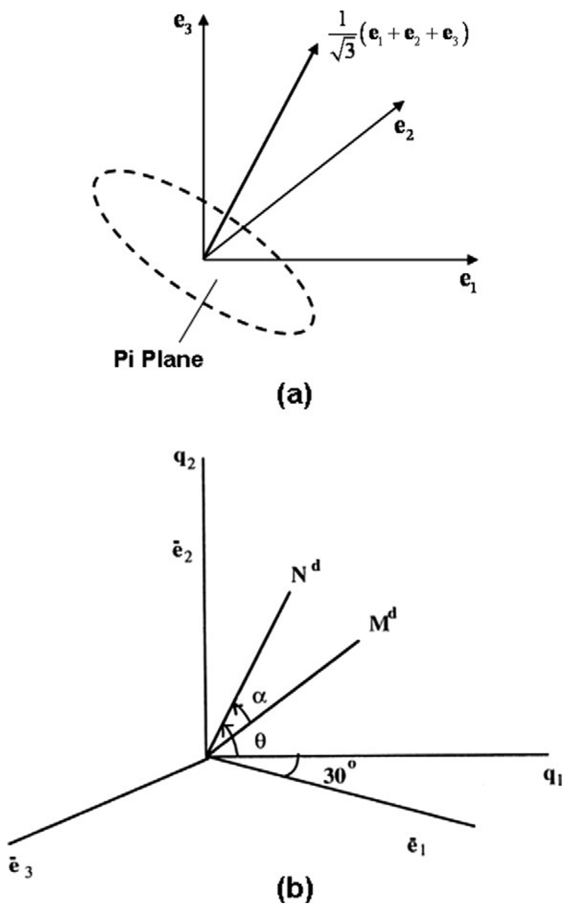

Fig. 1. Representation of nonassociativity: (a) orientation of the Pi plane for **M** and **N**, (b) projections of **M** and **N** in the Pi plane.

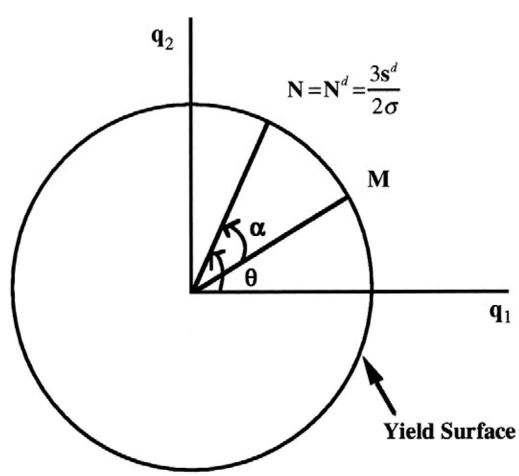

Fig. 2. Plastic flow direction, **M**, and normal to the von Mises yield surface, **N**, in the Pi plane.

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The principal directions of $\mathbf{M}$ have been assumed to be the same as those of $\mathbf{N}$: $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$; consequently, $\mathbf{M}$ is completely determined once $\mathbf{N}$ is known and the two parameters n and α are specified:

$$\mathbf{M} = \sum_{i=1}^{3} M_i \mathbf{e}_i \otimes \mathbf{e}_i. \tag{29}$$

Consider the special case where $\alpha = 0$, but $n \neq 1$. In this case, Eq. (28) reduces to

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \frac{1}{3}\begin{Bmatrix} 2N_1 - N_2 - N_3 \\ 2N_2 - N_3 - N_1 \\ 2N_3 - N_1 - N_2 \end{Bmatrix} + n\frac{N_1 + N_2 + N_3}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} N_1^d \\ N_2^d \\ N_3^d \end{Bmatrix} + n\frac{N_1 + N_2 + N_3}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}. \tag{30}$$

That is, $tr\mathbf{M} \neq tr\mathbf{N}$ but $\mathbf{M}^d = \mathbf{N}^d$, and deviatoric associativity is recovered. Furthermore, if $\alpha = 0$ and $n = 1$ then

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}, \tag{31}$$

or, $\mathbf{M} = \mathbf{N}$, and the associated flow rule is recovered.

4. Application to Mises plasticity with deviatoric nonassociativity

4.1. Initial comments

Since much research has been done on the bifurcation analysis of the elastic–plastic materials described by the plastic flow rules which are nonassociated in the spherical space but retaining associativity in the deviatoric space (e.g., Rudnicki and Rice, 1975; Peric et al., 1992; Larsson et al., 1993), here, the focus is on the influence of nonassociativity in the deviatoric space. Of particular interest is the influence of the nonassociativity parameter α on the failure angle and the critical plastic modulus, obtained by letting $n = 1$ and $\alpha \neq 0$. As an example, a bifurcation analysis is performed for a material described by the von Mises yield surface, but with a nonassociative flow rule in the deviatoric space.

4.2. Formulation

The von Mises yield surface is defined by

$$f(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = \bar{\sigma} - H(\bar{\varepsilon}^p) = \left[\frac{3}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d\right]^{1/2} - H(\bar{\varepsilon}^p), \tag{32}$$

where $\bar{\varepsilon}^p$ is the equivalent plastic strain defined by

$$\dot{\bar{\varepsilon}}^p = \left[\frac{2}{3}(\dot{\boldsymbol{\varepsilon}}^p)^d : (\dot{\boldsymbol{\varepsilon}}^p)^d\right]^{1/2} = \dot{\lambda}\left[\frac{2}{3}\mathbf{M}^d : \mathbf{M}^d\right]^{1/2}, \tag{33}$$

and $H(\bar{\varepsilon}^p)$ is the strain-hardening function. The normal to the yield surface is

$$\mathbf{N} = \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{3}{2}\frac{\boldsymbol{\sigma}^d}{\bar{\sigma}}, \quad (\mathbf{N} : \mathbf{N})^{1/2} = \left(\frac{3}{2}\frac{\boldsymbol{\sigma}^d}{\bar{\sigma}} : \frac{3}{2}\frac{\boldsymbol{\sigma}^d}{\bar{\sigma}}\right) = \left(\frac{3}{2}\right)^{1/2}. \tag{34}$$

Since $\mathbf{N}$ is proportional to $\boldsymbol{\sigma}^d$, the principal directions and the Pi plane for $\mathbf{N}$ and $\mathbf{M}$ are the same as those for the stress tensor, $\boldsymbol{\sigma}$. In this case, every angle θ represents a stress path. For example, $\theta = 0$ corresponds to pure shear; $\theta = -30^\circ$ triaxial tension and $\theta = 30^\circ$ triaxial tension, as shown in Fig. 1b.

A bifurcation analysis for associated flow rule subjected to the following standard loading paths have been performed by Schreyer and Neilsen (1996a,b). Here, we examine the effect of the degree of nonassociativity on the bifurcation prediction.

4.3. Restriction to plane stress

Consider the case of plane stress in the $\mathbf{e}_1$—$\mathbf{e}_2$ plane. Let $(\sigma_1, \sigma_2, 0)$ be the principal stresses. Then the principal values of the stress deviator, $\boldsymbol{\sigma}^d$, are $(2\sigma_1 - \sigma_2, 2\sigma_2 - \sigma_1, -\sigma_1 - \sigma_2)/3$. It follows from Eqs. (32) and (34) that $\bar{\sigma} = (\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2)$ and

$$\mathbf{N} = \frac{1}{2\bar{\sigma}}[(2\sigma_1 - \sigma_2)\mathbf{e}_1 \otimes \mathbf{e}_1 + (2\sigma_2 - \sigma_1)\mathbf{e}_2 \otimes \mathbf{e}_2 - (\sigma_1 + \sigma_2)\mathbf{e}_3 \otimes \mathbf{e}_3]. \tag{35}$$

The principal values of $\mathbf{N}$ are

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \frac{1}{2\bar{\sigma}}\begin{Bmatrix} 2\sigma_1 - \sigma_2 \\ 2\sigma_2 - \sigma_1 \\ -(\sigma_1 + \sigma_2) \end{Bmatrix}. \tag{36}$$

The influence of the degree of nonassociativity, α (as defined in Fig. 1b and Fig. 2), on the failure angles and the critical moduli for special cases are given next. In all examples, a value of 1/4 is selected for the Poisson's ratio of the material.

(i) Uniaxial stress

Let $\sigma_1 = \sigma > 0$, $\sigma_2 = 0$, a loading path defined as uniaxial tension. The normal to the failure plane and the critical modulus at which bifurcation first occurs are found. The axisymmetry of the problem makes the critical direction $\mathbf{n}$ (normal to the failure plane) nonunique in the $\mathbf{e}_2$—$\mathbf{e}_3$ plane. Fig. 3a shows a plot of the angle, β,

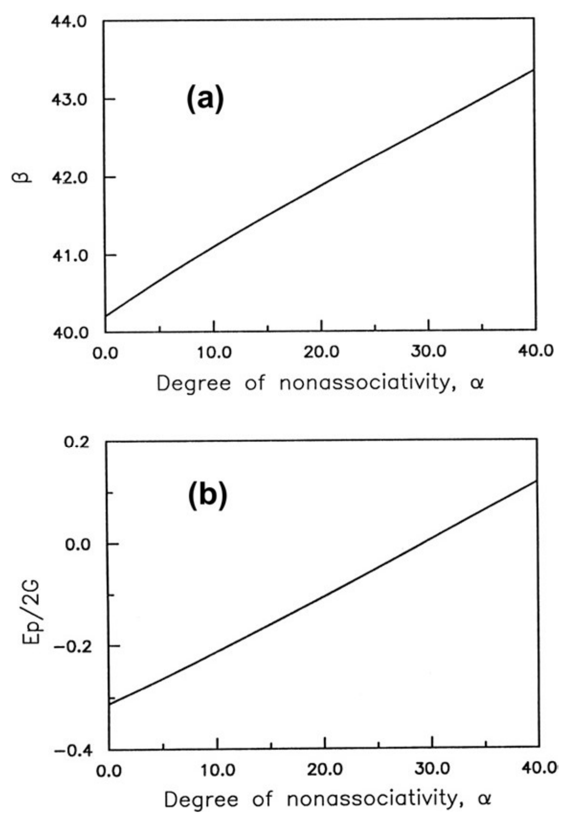

Fig. 3. Influence of degree of nonassociativity on the failure of a rod subjected to uniaxial tension (a) the failure angle, (b) the critical modulus.

made by **n** with respect to the direction of the applied stress, as a function of the degree of nonassociativity α. The plot starts with $\beta = 40^o$ for an associated flow rule ($\alpha = 0$), the result first given by Rudnicki and Rice (1975), and increases slowly with α. For a fairly large degree of nonassociativity ($\alpha = 40^\circ$), the failure angle β only increases about 3 degrees, which is less than 8% of the original angle corresponding to $\alpha = 0$. Fig. 3b plots the non-dimensional critical modulus ($E_p/2G$) with α, the degree of nonassociativity. The modulus starts at about -0.33 for an associated flow rule (again, the result first given by Rudnicki and Rice), which means bifurcation occurs only after the material enters the softening regime. The critical modulus increases with the degree of nonassociativity. For $\alpha < 30^\circ$, the modulus us still negative. The critical modulus reaches zero at $\alpha = 30^\circ$, which means that the first bifurcation coincides with the limit point for that particular degree of nonassociativity. For $\alpha > 30^\circ$, the critical modulus becomes positive which implies that with a large enough angle of nonassociativity, this material can bifurcate when it is still strain-hardening. Fig. 3a and b show that for $\alpha = 40^\circ$, although the failure angle increases less than 8% from the value for an associated flow rule, the critical modulus changes the sign and bifurcation occurs much sooner than if an associated flow rule is used.

(ii) Pure shear

Let $\sigma_1 = \sigma > 0$, $\sigma_2 = -\sigma$, describe a loading path that results in pure shear of the material (Fig. 4). The failure direction is in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane. Fig. 5a shows that the failure angle starts with $\beta = 45^\circ$ for an associated flow rule, and increases monotonically with the degree of nonassociativity. Similarly to the case of uniaxial stress, the failure angle β here is not very sensitive to α (it only increase about 3.3° for a change in α of 40°). Fig. 5b shows that the effect of α on the non-dimensional critical plastic modulus ($E_p/2G$). The critical modulus starts at zero for $\alpha = 0$, meaning that if the flow rule for the material is associated, then the material bifurcates at the peak in the stress–strain curve. With an increase in α, the value of $E_p/2G$ becomes positive, meaning the plate will bifurcate before the peak. For $\alpha = 40^\circ$, although the failure angle only increases less than 8%, the critical modulus reaches 2% of the elastic shear modulus, which could move the bifurcation point into the beginning of the strain-hardening regime. Both uniaxial stress and pure shear loading paths show that while the failure angles do not change much with the introduction of the nonassociativity, the critical plastic moduli (or failure loads) do show strong dependence on α. Earlier bifurcation will be predicted if an associated flow rule is replaced by a nonassociated one.

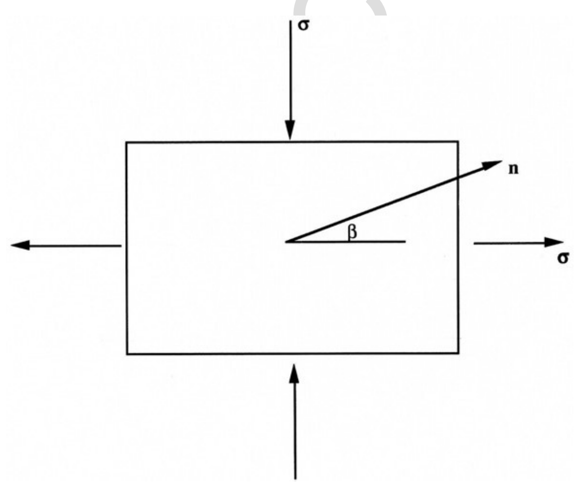

Fig. 4. A state of pure shear.

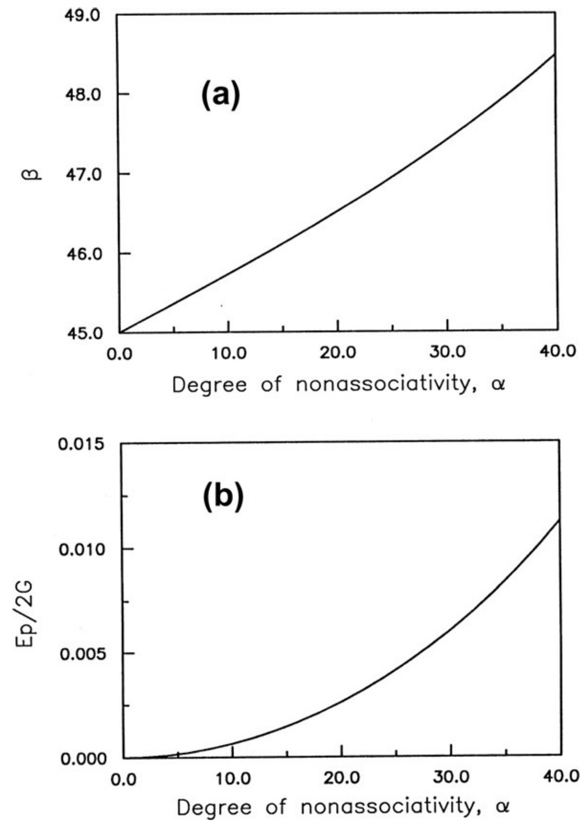

Fig. 5. Influence of degree of nonassociativity on the failure of material subjected to pure shear (a) the failure angle, (b) the critical modulus.

(iii) Equal biaxial tension

Let $\sigma_1 = \sigma_2 = \sigma$. Since both the yield surface and the plastic flow rule under consideration are insensitive to hydrostatic pressure, the result is identical to that of uniaxial stress with the exception that the failure angle β is now with respect to the $\mathbf{e}_3$ direction.

(iv) Unequal biaxial tension

Let $\sigma_1 = 2\sigma$, $\sigma_2 = \sigma$, a stress state often encountered in a cylindrical pressure vessel with $\mathbf{e}_1$ along the circumferential (hoop) direction and $\mathbf{e}_2$ along the axis of the cylinder. The result is identical to that of pure shear with the exception that now the failure direction lies in the $\mathbf{e}_1$–$\mathbf{e}_3$ plane as opposed to the $\mathbf{e}_1$–$\mathbf{e}_2$ plane for the pure shear case (ii) discussed earlier.

(v) Triaxial compression

Another common test is that of triaxial compression, which is not a case of plane stress, whereby a longitudinal compressive stress is superposed on a state of uniform pressure. However, since **N** and **M** are insensitive to hydrostatic pressure for the model considered here, the result is identical to that for uniaxial stress.

4.4. Discussion of results

To illustrate the important implications of this section, consider the sketch of a generic stress–strain curve in Fig. 6. For uniaxial tension (Fig. 6a), the point of bifurcation for an associated law occurs well into the softening regime as indicated by **A**. For $\alpha = 40^\circ$, the bifurcation point is denoted as $\mathbf{A}'$, while the materials is still in the strain-hardening. Similar points for pure shear (Fig. 6b) are denoted by **B** (the peak) and $\mathbf{B}'$, respectively. In both cases, with $\alpha = 40^\circ$ bifurcation points ($\mathbf{A}'$ and $\mathbf{B}'$) move into the hardening regime.

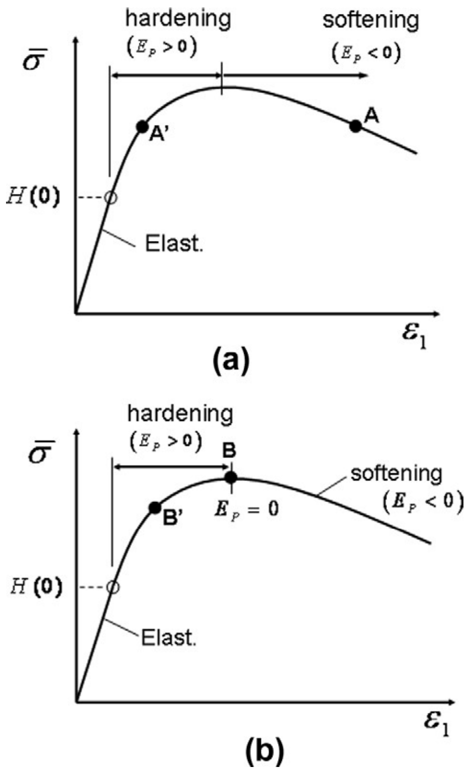

Fig. 6. Sketch of a generic stress–strain curve illustrating the effect of deviatoric nonassociativity on the onset of bifurcation.

These same trends hold for nonassociativity in the spherical space (Rudnicki and Rice, 1975). However, there are some situations where deviatoric nonassociativity may be the dominant aspect, and, therefore, it is possible to have a bifurcation much earlier than previously thought. More importantly, deviatoric nonassociativity may be physically relevant for a number of significant cases.

5. Summary and conclusions

The discontinuous bifurcation analysis based on the loss of ellipticity condition is performed on elastic–plastic materials where the flow rule can be nonassociated in both the spherical and the deviatoric spaces. A particular form of nonassociativity in the deviatoric space is introduced, where the projections of the plastic flow direction (**M**) and the normal to the yield surface (**N**) are assumed to have the same length but the projection of **M** lags that of **N** by an angle. With this assumption, nonassociativity in the deviatoric space is completely characterized by the angle.

It is shown that even for the simple yield surface of von Mises, nonassociativity in the deviatoric space can lead to a bifurcation for a load parameter that is significantly lower than the value predicted with an associated flow rule. Furthermore, the degree of nonassociativity in the deviatoric space seems to have only a modest effect on the orientation of the bifurcation (failure) plane.

In our numerical calculations, we have only considered the case where the projection of **M** lags that of **N** ($\alpha > 0$). For a non-radial loading path, depending on the direction of the stress increment with respective to the normal to the yield surface (**N**), the projection of **M** may lead **N**, rather than lags it. We expect similar results (i.e., lowering the critical load parameter) hold for $\alpha < 0$, but that remains to be verified.

6. Uncited reference

Runesson et al. (1991).

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