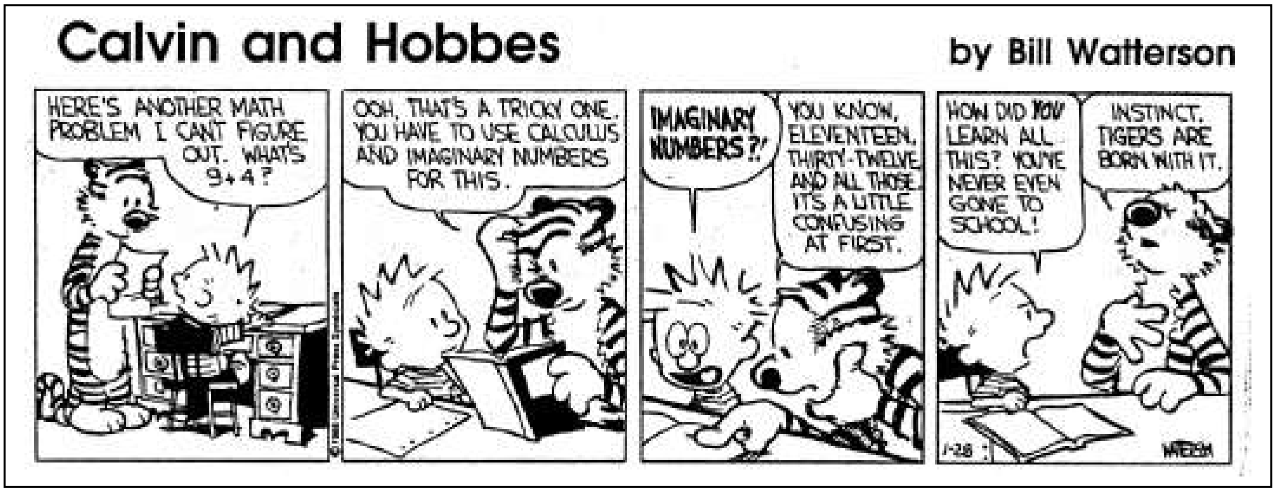
Complex Numbers



In spite of Calvin's discomfiture, imaginary numbers (a subset of the set of *complex numbers*) exist and are invaluable in mathematics, engineering, and science. In fact, in certain fields, such as electrical engineering, aeronautical engineering and quantum mechanics, progress has been critically dependent on complex numbers and their behavior. In the context of mathematical biology, a number of models we'll be working with can yield solutions that involve complex numbers, so we need to refresh our memories about complex numbers and how to work with them.

The Quadratic Formula and Complex Numbers

Aside from allowing us to solve difficult problems such as $9+4=??$, probably the most frequent situation in which most of us have encountered complex numbers has been when finding the roots of quadratic equations of the form

$$f(x) = Ax^2 + Bx + C = 0 \qquad \text{Equation 1}$$

You may recall from your high-school algebra course that the roots of Equation 1 are the values of *x* for which the equation is exactly equal to zero, and that we can easily solve for the roots of Equation 1 by means of the quadratic formula, which usually takes the form

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

where *x* represents a root of Equation 1. Note that the ± sign tells us that quadratic equations will have two roots that differ in value by a quantity equal to the value of $\frac{\sqrt{B^2 - 4AC}}{2A}$ (which may equal zero). Let's work with Equation 1 and set $A = C = 2$. This leads to the following expression for the quadratic formula

$$x = \frac{-B \pm \sqrt{B^2 - 16}}{4}$$

which we will now solve for $B = -5 \ldots 5$. The table on the next page gives the results of these calculations.

B	$B^2 - 4AC$	$x = \dfrac{-B \pm \sqrt{B^2 - 16}}{4}$
5	9	$-0.5,\ -2$
4	0	$-1,\ -1$
3	-7	$-0.75 \pm \dfrac{\sqrt{-7}}{4} = -0.75 \pm 0.6614\sqrt{-1}$
2	-12	$-0.5 \pm 0.8660\sqrt{-1}$
1	-15	$-0.25 \pm 0.9682\sqrt{-1}$
0	-16	$\pm\sqrt{-1}$
-1	-15	$0.25 \pm 0.9682\sqrt{-1}$
-2	-12	$0.5 \pm 0.8660\sqrt{-1}$
-3	-7	$0.75 \pm 0.6614\sqrt{-1}$
-4	0	$-1,\ -1$
-5	9	$-0.5,\ -2$

We thus see that a quadratic equation has two roots (which may, depending on the values of *A*, *B*, and *C*, be equal to each other). However, the roots for $B = -3, -2, \ldots 2,\ 3$ probably look a little strange, involving that $\sqrt{-1}$. We're about to enter the strange and fabulous world of complex numbers

Imaginary and Complex Numbers

We first focus on the entry for *B* = 0, where the result is $\pm\sqrt{-1}$. Of course, there is no *real* number whose square is -1, so the result is referred to as an *imaginary number*. By convention, $\sqrt{-1}$ is designated by the letter *i*. All imaginary numbers may then be represented as *bi*, where *b* is any real number. Our quadratic equation calculations for *B* = 0 thus become ± *i*.

But, what if we let $B = -3, -2, \ldots 2,\ 3, \quad B \neq 0$? Here the results are a bit more complicated, involving a combination of real and imaginary numbers. Such numbers are referred to as *complex numbers*, and are usually represented as *a* + *bi*, where *a* and *b* are real numbers that may take on any value between $-\infty$ and $+\infty$. A few points to note here:

1. If *b* = 0, you have the real number *a*.
2. If *a* = 0, the number *bi* is said to be *pure imaginary*, or, more simply, *imaginary*.
3. If both $a, b \neq 0$, then the number is said to be *complex*. In this case
   - *a* is referred to as the *real part* of the complex number, and is represented as Re(*x*).
   - *b* is termed the *imaginary part* of the complex number, and is represented by Im(*x*).
4. In the context of complex numbers, then, the roots of a quadratic equation may be:
   - real,
   - pure imaginary, or
   - complex, in which case they occur as the *complex conjugates*, $a + bi$ and $a - bi$.
5. Finally, points #1 and #2 show that the real and imaginary numbers are subsets of the complex numbers, suggesting that the basic mathematical operations of addition,

subtraction, multiplication, division, exponentiation, etc. can be applied to imaginary and complex numbers. This turns out to be the case, albeit some modifications are necessary.

Addition and Subtraction of Complex Numbers

Addition and subtraction of complex numbers are completely transparent, but serve to illustrate an important technique that will prove useful to us later on. The formula for carrying out addition of complex numbers is simply

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

where $a\ldots d$ are real numbers. The corresponding formula for subtraction of complex numbers is

$$(a+bi)-(c+di)=(a-c)+(b-d)i$$

Note what was done each: *the real and imaginary parts of the two complex numbers were grouped*, after which summation was carried out separately within each group, yielding another complex number with real part equal to $(a\pm c)$ and imaginary part equal to $(b\pm d)$. Also note that if $b = d = 0$, we're just adding or subtracting real numbers.

Multiplication and Division of Complex Numbers

Multiplication of complex numbers is straightforward, and should look familiar to you from your algebra days:

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+bdi^2\\ &= (ac+bdi^2)+(ad+bc)i\\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

with the $-bd$ coming from the fact that $i^2=-1$. Note that we again grouped like terms (real and imaginary) to obtain the final result.

Complex division is similarly straightforward. Given the complex division problem

$$\frac{(a+bi)}{(c+di)}$$

we first note that we if we multiply both the numerator and denominator by $c-di$:

$$\frac{(a+bi)}{(c+di)}\frac{(c-di)}{(c-di)}=\frac{(ac-bdi^2+bci-adi)}{(c^2-d^2i^2)}$$

we can get rid of that *i* in the denominator. (what term do we apply to the combination of $c+di$ and $c-di$?) Note the value of the original quotient is unchanged since $\frac{(c-di)}{(c-di)}=1$.We then group terms to obtain

$$\frac{ac+bd+(bc-ad)i}{(c^2+d^2)}$$

if $c^2+d^2\neq 0$. We can then immediately derive the final result by *separating* the real and imaginary terms to yield the final result:

$$\frac{(a+bi)}{(c+di)}=\frac{ac+bd}{(c^2+d^2)}+\frac{(bc-ad)}{(c^2+d^2)}i$$

Like complex addition and subtraction, complex multiplication and division yield complex numbers of the standard form, a real part and an imaginary part consisting of a real number multiplied by *i*. Also note that if $b = d = 0$ the above problems reduce to multiplication or division of real numbers.

I don't expect you to commit to memory the preceding material on complex arithmetic. I *do*, however, want you to keep in mind the techniques of grouping terms and multiplying $\frac{(a+bi)}{(c+di)}$ by $\frac{(c-di)}{(c-di)}$ to 'clean up' the denominator (this is frequently referred to in textbooks as *rationalizing the denominator*, although that's not strictly speaking correct usage of the term). Both are useful algebraic 'tricks' that will come into play a number of times during lecture and lab sessions next semester.

Exponents Involving Complex Numbers

We now arrive at the reason for our foray into the world of complex numbers. Next semester, you will encounter a number of simple models that can yield complex solutions of the form:

$$y(t) = f\left(e^{(a \pm bi)\,t}\right) \qquad \text{Equation 2}$$

where *e* is the base of natural logarithms (= 2.71828…) and $a \pm bi$ is a complex conjugate pair. How on earth can we generate something meaningful out of $e^{(a \pm bi)t}$, a number raised to a power that involves complex numbers? That seems mind-breakingly absurd in the extreme.

Well, in 1748 Leonhard Euler, one of history's pre-eminent mathematicians, showed us the way. First, we take advantage of the fact that, as with real numbers, an exponential term involving a complex sum can be decomposed into the product of two exponential terms. For example, if we start with the $a + bi$ member of the conjugate pair, we obtain:

$$e^{(a+bi)\,t} = e^{at} e^{ibt} \qquad \text{Equation 3}$$

That looks promising, since *a* is a real number and we know how to deal with e^{at}. However, there's still that pesky e^{ibt} term remaining to be dealt with, so it might not seem as though we've gained anything at all. But, we then recall that powers of *e* can be represented by an infinite series:

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}, \quad n \le \infty$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

where $n! = 1 \cdot 2 \cdot 3 \cdots (n-2) \cdot (n-1) \cdot n$ (i.e., *n* factorial). If we substitute *ibt* for *x* in Equation 3 and recall that $i^2 = -1$, $i^4 = 1$, $i^6 = -1$, etc., we can accomplish the following:

$$e^{ibt} = 1 + ibt + \frac{(ibt)^2}{2!} + \frac{(ibt)^3}{3!} + \frac{(ibt)^4}{4!} + \frac{(ibt)^5}{5!} + \frac{(ibt)^6}{6!} + \frac{(ibt)^7}{7!} + \cdots + \frac{(ibt)^n}{n!}$$

$$= 1 + ibt + i^2 \frac{(bt)^2}{2!} + i^2 \cdot i \frac{(bt)^3}{3!} + i^4 \frac{(bt)^4}{4!} + i^4 \cdot i \frac{(bt)^5}{5!} + i^6 \frac{(bt)^6}{6!} + i^6 \cdot i \frac{(bt)^7}{7!} + \cdots$$

$$= 1 + ibt - \frac{(bt)^2}{2!} - i\frac{(bt)^3}{3!} + \frac{(bt)^4}{4!} + i\frac{(bt)^5}{5!} - \frac{(bt)^6}{6!} - i\frac{(bt)^7}{7!} + \cdots.$$

Next, taking a cue from complex arithmetic, we group real and imaginary terms to obtain:

$$e^{ibt} = 1 - \frac{(bt)^2}{2!} + \frac{(bt)^4}{4!} - \frac{(bt)^6}{6!} \cdots + ibt - i\frac{(bt)^3}{3!} + i\frac{(bt)^5}{5!} - i\frac{(bt)^7}{7!} \cdots$$

$$= \left(1 - \frac{(bt)^2}{2!} + \frac{(bt)^4}{4!} - \frac{(bt)^6}{6!} \cdots\right) + i\left(bt - \frac{(bt)^3}{3!} + \frac{(bt)^5}{5!} - i\frac{(bt)^7}{7!} \cdots\right) \qquad \text{Equation 4}$$

which may not seem like much of a gain, since it's still a complex number. Fortunately – and here's the cool part – Euler recognized that, like e^x, $\cos x$ and $\sin x$ also have infinite series equivalents:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{Equation 5a}$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \qquad \text{Equation 5b}$$

Taken together, Equation 5a and 5b allow us to rewrite Equation 4 as the sum of sine and cosine terms:

$$e^{ibt} = \underbrace{\left(1 - \frac{(bt)^2}{2!} + \frac{(bt)^4}{4!} - \frac{(bt)^6}{6!} + \cdots\right)}_{\cos bt} + i\underbrace{\left(bt - \frac{(bt)^3}{3!} + \frac{(bt)^5}{5!} - \frac{(bt)^7}{7!} + \cdots\right)}_{\sin bt}$$

$$= \cos bt + i \sin bt$$

This extremely important result is simply a modification of *Euler's formula* ($e^{ix} = \cos x + i \sin x$), "the most remarkable formula in mathematics" (R. Feynman, 1977), with *ibt* substituted for *x*.

If we had instead started our derivation with the $a - bi$ conjugate, we would have arrived at

$$e^{-ibt} = \cos bt - i \sin bt$$

meaning the two solutions for Equation 2 are

$$y(t) = e^{(a+bi)t} = e^{at}(\cos bt + i \sin bt)$$

and

$$y(t) = e^{(a+bi)t} = e^{at}(\cos bt - i \sin bt)$$

The next step, which we won't detail, combines those two solutions and leads to the ultimate solution:

$$\boxed{y(t) = e^{at}(c_1 \cos bt + c_2 \sin bt)} \qquad \text{Equation 6}$$

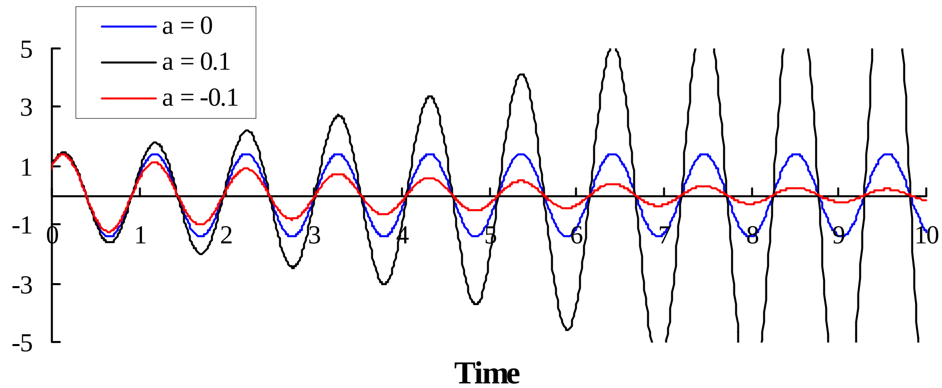
where *a* and *b* are the real components of the complex exponent in Equation 2, and c_1 and c_2 are constants whose values are determined by the initial conditions specified in the model. The end result is that a complex exponential function (Equation 2) has been converted into a purely real-valued function that's easily interpreted. Thank you, Professor Euler!

What I Want You to Take From the Preceding

The preceding may seem a bit much, but I believe it's important to expose you to the underlying mathematics of topics we'll be studying. What I want you to remember is a little less intense:

Equations involving a complex exponential can be represented as *real* sine and cosine functions. Because sine and cosine functions are periodic, such equations will exhibit periodic (cyclic) behavior. Since a number of the models we'll be working with in this course can, under realistic conditions, yield solutions of complex exponential form, *those models will exhibit periodic behavior*.

We will delve extensively into periodic functions and the ramifications of Equation 6 during the upcoming semester, because models that lead to solutions of the form represented by Equation 2 have important applications in a wide variety of biological models. For now, we'll content ourselves with a look at graphs of Equation 6 for $c_1 = c_2 = 1$, $b = 6$, and a = -0.1, 0, or 0.1:



The important features to note in this figure are that (i) the solutions oscillate, and (ii) the sign of *a* determines whether the oscillations' amplitude decreases, remains constant, or increases with time.

Periodic Functions

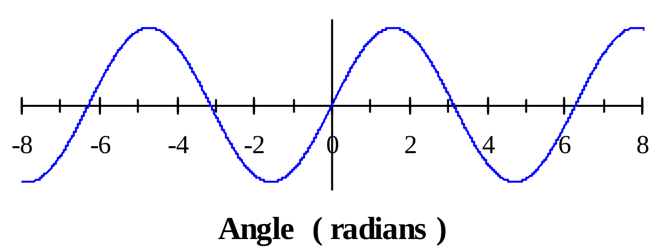
As an aside, it's worth noting that the term "periodic" has a precise definition in mathematics. A function $f(x)$ is said to be periodic if and only if there exists some interval, *L*, such that

$$f(x_{i+L}) = f(x_i)$$

and

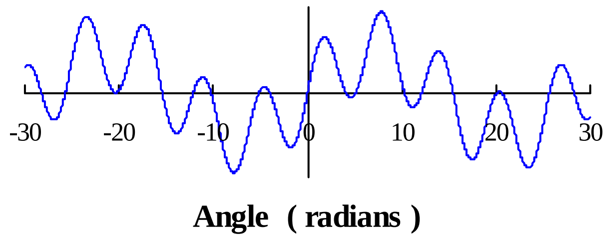
$$f^n(x_{i+L}) = f^n(x_i), \quad n = 1,2,3,\ldots$$

where $f^n(x)$ represents the *n*th-order derivative of $f(x)$. In others words, in order for a function to be truly periodic, both the *value* of the function *and* of all *derivatives* of the function must have the same value at beginning of the interval and at the end. We can illustrate this with a couple examples. First, consider the sine function:



Note that any given value of the sine function repeats a number of times over the interval from – 8 to +8, but that the value *and* the derivatives of the function repeat themselves *only* at intervals of $2\pi\ (\cong 6.28)$. The sine function is therefore termed "2π periodic".

Now, try to determine the period of the following function:



Can the period be determined from the data given to you?