UNM - PNM STATEWIDE MATHEMATICS CONTEST XLI

November 8, 2008   First Round   Three Hours

1) Our number system is base 10. Every number can be written as a sum of a digit from 0 to 9 times a power of 10, i.e. $239 = 2 \cdot 10^2 + 3 \cdot 10^1 + 9 \cdot 10^0$. In base n, a number is written as a number in the set $\{0, 1, \ldots, n-1\}$ times a power of n. How would you write our current year, 2008, in the base 2?

2) How can you assign numbers 1, 3, 5, 7 and 9 to c, o, k, i, e so that the numerical equivalent to cookie is divisible by 3?

3) Equilateral triangles are inscribed in and circumscribed outside a circle. If the area of the larger triangle is 15 square units, what is the area of the smaller triangle? Hint: In a right triangle with angles 30°, 60° the length of the side across the angle of 30° is one-half the length of the hypotenuse.

4) If $x + y + z = 5$, $xy + xz + yz = 6$ and $xyz = 4$, what are
   a) $x^2 + y^2 + z^2$?   b) $x^3 + y^3 + z^3$?

5) Suppose two circles of diameter one are tangent at the point $(\frac{1}{2}, \frac{1}{2})$ and the first circle is tangent to the x-axis at $(0, 0)$ and the second circle is tangent to the x-axis at the point $(1, 0)$. What is the circle with the largest area that can be inscribed in the region bounded by the two circles and the x-axis?
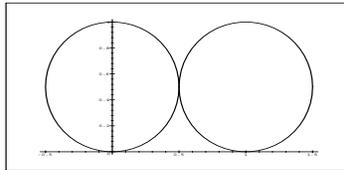
6) Let f be a function defined on the integers as follows: $f(n) = n - 5$ for $n > 50$ and $f(n) = f(f(n+6))$ for $n \le 50$. What is the range of f?

7) We start with the rectangle with vertices $(0, 0)$, $(3, 0)$, $(0, 4)$ and $(3, 4)$ and rotate it $90°$ clockwise about the vertex $(3, 0)$ forming a new rectangle. We again rotate this new rectangle $90°$ about the vertex $(7, 0)$ forming a new rectangle. Twice more we rotate $90°$ clockwise about the vertices $(10, 0)$ and $(14, 0)$. Find the area under the path formed by the movement of the point $(0, 1)$ under these rotations.

8) A group of 100 tourists arrived in a city. Of these tourists, 15 knew neither French nor Italian, 65 knew French, and 77 knew Italian. How many tourists knew both French and Italian?

9) Suppose $x^2 - 6x + k$ and $x^2 - 10x + 2k$ have a common root. What must k be?

10) A bank ATM machine has $500 in one dollar bills. You are allowed to withdraw $300 or deposit $198. What is the maximum amount you can withdraw from this ATM machine? You can do any sequence of withdrawals and deposits (of the allowed type).