December 2008 Problems

Please send your solutions or questions to Janet Vassilev (jvassil@math.unm.edu) or Dimiter Vassilev (vassilev@math.unm.edu). We are looking forward to hearing from you.

1. (Wilson's theorem) As customary, given three integers m, n and r, $n > 0$, we write

$$m \equiv r \; mod \; n$$

to indicate that the integer $m - r$ is a multiple of n. In other words, the remainder of the division of $m - r$ by n is 0. For example, $9 \equiv 1 \; mod \; 4$ since $9 = 2 \cdot 4 + 1$, while $49 \equiv 4 \; mod \; 5$. Also, we denote with $m!$ (m factorial) the product $1 \cdot 2 \cdot 3 \cdot \cdots \cdot m$, i.e., the product of the natural numbers from 1 to m.

a) Let x and p be integers with p a prime number. By observing that, trivially, $x^2 - 1 \equiv (x-1)(x+1) \; mod \; p$, show that the only solutions of $x^2 \equiv 1 \; mod \; p$ with $x \in \{0,\, 1,\, 2, \ldots, p-1\}$, the set of remainders when you divide by p, are $x = 1$ and $x = p - 1$.

b) Conclude by pairing solutions of $xy \equiv 1 \; mod \; p$ that

$$(p-1)! \equiv -1 \; mod \; p.$$

c) Show that, $(n-1)! \equiv -1 \; mod \; n$ if and only if n is a prime number.

d) Let n be a natural number. Consider the remainders of the division by n, i.e., the set $\mathbb{Z}_n^* = \{0,\, 1,\, 2, \ldots, n-1\}$. We call a remainder $x \in \mathbb{Z}_n^*$ a unit if there is a $y \in \mathbb{Z}_n^*$ such that $xy \equiv 1 \; mod \; n$. Show that the product of all units has remainder $+1$ or -1 when divided by n, i.e., the product of all units is a solution of $X \equiv \pm 1 \; mod \; n$.

Note: You might be interested in learning more about $mod \; n$ arithmetic, in which case you can look up a book in number theory. You should start with proving that given two integers m and n, $n > 0$, there is exactly one natural number r among $\{0,\, 1, \ldots, n-1\}$ such that $m \equiv r \; mod \; n$. This number is called the remainder of the division of m by n.

2. Show that the function

$$f(x,y) = \frac{y-1}{2}\left[|B^2 - 1| - (B^2 - 1)\right] + 2,$$

where $B = x(y+1) - (y! + 1)$, x and y natural numbers, generates only prime numbers and all of them, and each odd prime number exactly once. Hint: Use Wilson's theorem.

3. Show that given n points in the plane, $n \geq 3$, which do not lie on the same line (i.e., the points are not all collinear), it is possible to draw a line which passes through exactly two of the given n points.

4. A graph in the plane is a set of points (called vertices) some of which are connected with segments (called edges). A graph is called *planar* if it can be drawn in the plane without intersecting any edges. A graph is called *connected* if you can use the given edges to travel from any vertex to any other vertex.

Suppose you are given a connected planar graph. The edges split the plane into regions, one of them unbounded, which are called faces. Prove Euler's formula

$$v - e + f = 2,$$

where v is the number of vertices, e is the number of edges, and f is the number of faces in the graph.

5. Let us color red all points in the plane with integer coordinates. Show that the area of any triangle with red vertices and no red points inside or on its boundary, except the vertices, has area equal to 1/2.