January 2009 Problems

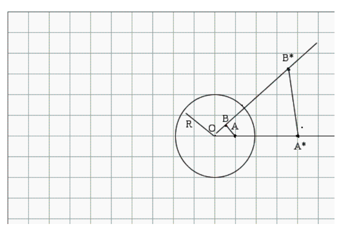
Please send your solutions or questions to Janet Vassilev (jvassil@math.unm.edu) or Dimiter Vassilev (vassilev@math.unm.edu). We are looking forward to hearing from you.

1. Most of you are probably familiar with symmetry with respect to a fixed line in the plane. The symmetric image of a triangle or a circle in the plane is a congruent (identical) triangle or circle in the plane. Give a formal proof that if A and B are two points in the plane and l is a fixed line, then the length $|AB|$ of the segment AB is the same as the length $|A'B'|$ of the segment $A'B'$, where A' and B' are the symmetric images with respect to l, correspondingly, of A and B.

   Inversion in the plane. *Let O be a fixed point in the plane and R a positive number $R > 0$. An inversion with respect to the circle of radius R and center O (symmetry with respect to a circle) is defined as follows. Let A be a point in the plane different from O. The symmetric image of A, called also the inversion of A, with respect to the fixed circle is the point A^* on the axis starting at O and passing through A such that $|OA| \cdot |OA^*| = R^2$.*

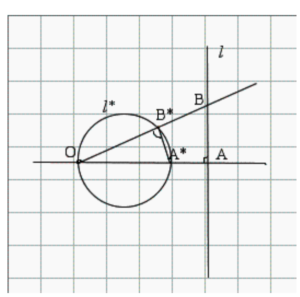
   The next set of problems concern the properties of the inversion with respect to a circle. In all of the following problems, we let O be a fixed point in the plane, R be a positive number, and c_R be the circle of radius R and center O. For given points A, B, C etc., all different from O, let A^*, B^*, C^* be the corresponding images under the inversion with respect to the circle c_R.

2. a) Show that if A is on the circle c_R then $A^* = A$, i.e., the circle c_R does not "move" under the inversion with respect to itself.
   b) Show that if A is inside the circle c_R then A^* is outside the circle.
   c) Show that $(A^*)^* = A$. In other words, the symmetric image of the symmetric image with respect to c_R of a point A is the point A.
   d) Show that the triangle $\triangle OAB$ is similar to the triangle $\triangle OB^*A^*$, i.e., $\triangle OAB \approx \triangle OB^*A^*$.



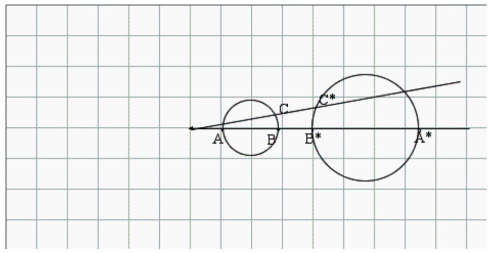
3. Show that the image of a line passing through the center of inversion O is the same line (the point O is excluded).

4. a) Show that the under the inversion the image of a line l not passing through the center of inversion is a circle passing through the center of inversion with the point O removed from this circle. To see this, take the point A on the line l which is closest to O. Consider the circle c with diameter OA^*. Show that if B is point on l then B^* is on this circle. Hint: What characterizes all triangles inscribed in c having the diameter OA^* as one side? Use that $\triangle OAB \approx \triangle OB^*A^*$.

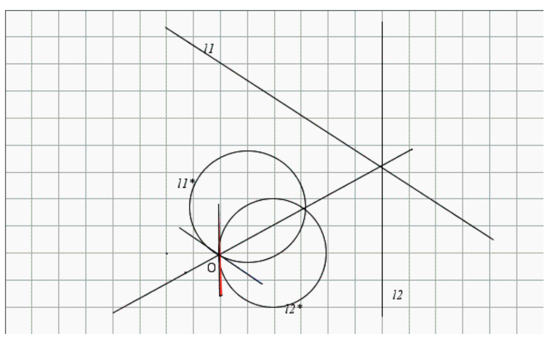


b) Show that under the inversion the image of a circle passing through the center of inversion O is a line not passing through O. This line is perpendicular to the diameter of the circle through O. Hint: Use part a).

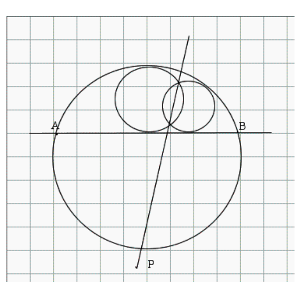
c) Show that the image under the inversion of a circle c not passing through the center of inversion O is a circle c^*(not passing through O). Hint: Consider the line through O and the center of c. This defines two diametrically opposed points A and B on c. Let c' be the circle with diameter A^*B^*. Show that $c' = c^*$. The hint of a) is relevant here as well.



5. Show that the if l_1 and l_2 are two lines in the plane, then the smallest angle between their images l_1^* and l_2^* under the considered inversion is the same as the smallest angle between the lines l_1 and l_2. Note that given two circles, which intersect at a point P, the smallest angle between the two circles at P is defined as the smallest angle between the tangent lines to each of the circles at P. With this definition, we can claim that the inversion preserves the angle between intersecting circles as well. Hint: To answer the question you will have to eventually consider only lines.



6. Let c be a circle and AB a chord on the circle (so A and B are two point on the circle). Consider the region which is the smaller of the two parts in which AB divides the disk with boundary c. Show that there is a point P on the circle c such that the line passing through the intersection points of any two circles inscribed in the defined region passes also through P (so the same point P is on any such line).



We can define similarly inversion in the three dimensional space. Using that inversion we can map a sphere into a plane (again preserving the angles).