## February 2009 Problems

*Please send your solutions or questions to Janet Vassilev (jvassil@math.unm.edu) or Dimiter Vassilev (vassilev@math.unm.edu). We are looking forward to hearing from you.*

Following are the 2008-2009 UNM-PNM 2nd round problems.

1. An equilateral triangle is inscribed in a circle which is circumscribed by a square. This square is inscribed in a circle which is circumscribed by a regular hexagon. If the area of the equilateral triangle is 1, what is the area of the hexagon?
2. Suppose you have a piece of paper which you cut into either four or sixteen pieces. After that you cut again some of the pieces into either four or sixteen smaller pieces. Suppose you have nothing else to do, so you keep repeating this procedure cutting some of the pieces into either four or sixteen smaller pieces. Can you end up with 2009 pieces at some stage of your cutting process?
3. A group of 200 high school students visited the University of New Mexico for Lobo Day. The students could participate in at most two of the following three workshops: 1) Algorithms, 2) Bioinformatics and 3) Coding. Suppose 100 students participated in Algorithms, 90 participated in Bioinformatic, 80 participated in Coding.
   (a) If 87 participated in both Algorithms and Bioinformatics or both Algorithms and Coding. How many students participated in both Bioinformatics and Coding?
   (b) If in addition 24 participated in Coding but not in Algorithms or Bioinformatics, how many participated in both Algorithms and Bioinformatics?
4. Find the smallest $N$ such that $\frac{1}{3}N$ is a perfect cube, $\frac{1}{7}N$ is a perfect seventh power and $\frac{1}{8}N$ is a perfect eighth power.
5. Determine the right triangle of smallest perimeter with integer sides, which has the property that the area is equal to three times the perimeter.
6. 2009 lines are drawn in the plane such that:
   - No two lines are parallel.
   - At all points of intersection, at least three lines meet.

   Show that all the lines go through one point.
7. The hat game is a collaborative game played by a team of 3 players. Either a red or a blue hat is put on each player's head. The players can see the other players hat colors, but cannot see and do not know the color of the hat on their heads. Each player must either guess the hat color on his/her head or pass. The team wins if every player who does not pass guesses the correct hat color and at least one player does not pass (i.e. makes a guess). Before playing the team can determine a strategy. Is there a strategy to win more than 50 percent of the time? If so, what is the strategy and what is the probability that the team will win?
8. Suppose the beam of each of four projectors lights an oval-shaped area on the stage of a theater. Show that if the spotlights of every three of the given four projectors overlap somewhere, then there is a place which is in the spotlight of all of the projectors.
9. Let $a$ be a non-negative number, i.e., $a \geq 0$. Define successively an infinite sequence of non-negative numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ by letting $a_1 = a$ and then using the formula

   $$a_{n+1} = \frac{1}{2}(a_n^2 + 1)$$

   for $n = 2, 3, \ldots$ ( $n$ runs through all positive integer numbers).

(a) Show that if $0 \leq a < 1$ then all of the numbers in the sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

are less than one.

(b) Show that if $a > 1$ then the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ has arbitrarily large numbers, i.e., given any number $M$ there is at least one number among $a_1, a_2, a_3, \ldots, a_n, \ldots$ which is larger than $M$.

10. Let $A$ be the only common point of two disks, not necessarily of the same radius, with boundaries the circles $c$ and $c'$. Suppose we draw two lines $l$ and $l'$ which are tangent to each of the circles. Let $B$ and $C$ be the points of contact of the line $l$, correspondingly, with $c$ and $c'$. Similarly, let $B'$ and $C'$ be the points of contact of the line $l'$, correspondingly, with $c$ and $c'$. Show that the circles circumscribed around the triangles $\triangle ABC$ and $\triangle AB'C'$ are tangent to each other.