October 2009 Problems

Please send your solutions or questions to Janet Vassilev (jvassil@math.unm.edu) or Dimiter Vassilev (vassilev@math.unm.edu). We are looking forward to hearing from you.

1) Prove that the polynomial $f(x) = x^3 - ax + 2$ has three integer roots if and only if $a = 3$. As usual, a root of a polynomial f is a number c such that $f(c) = 0$. A root c is said to be an integer root if c is an integer number.

2) Let f be a polynomial of the variable x with integer coefficients.

(a) Show that if f takes the value 1 for some four different integer values c_1, c_2, c_3, c_4 of x then f cannot take the value -1 at any integer number.

(b) Show that if f takes the value 1 for some three different integer values c_1, c_2, c_3 of x then f can take the value -1 for at most one integer number.

(c) Show that if f takes the value 1 for two different integer values c_1, c_2 of x then f can take the value -1 for at most two integer numbers.

3) If we remove 90% of the sugar from a "regular" cup of chocolate milk, how many of the reduced sugar cups do you have to drink in order to consume the same amount of sugar as you would consume drinking a "regular" cup of chocolate milk?

4) (a) Let $f(x)$ be a polynomial of degree 4, i.e., $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + +a_1x + a_0$, where a_0, a_1, a_2, a_3, a_4 are numbers and x is an independent variable. Suppose that you can factor $f(x)$ in the form

$$f(x) = a_4(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

for some numbers x_1, x_2, x_3, x_4 (which are thus the roots of the polynomial). Show that the following identities hold true

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= -a_3/a_4 \\ x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 &= a_2/a_4 \\ x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4 &= -a_1/a_4 \\ x_1x_2x_3x_4 &= a_0/a_4. \end{aligned}$$

(b) What do you obtain if you reason as in part (a) and f is a polynomial of degree two, three, five, or any natural number n?

5) A circle of radius R is divided in three arcs using the points A, B and C in such a way that the ratios of the lengths of these arcs are like $3 : 4 : 5$. Find the area of the triangle formed by the tangents to the circle at the given points.