

DECEMBER 2009 PROBLEMS

Please send your solutions or questions to Janet Vassilev ([jvassil@math.unm.edu](mailto:jvassil@math.unm.edu)) or Dimiter Vassilev ([vassilev@math.unm.edu](mailto:vassilev@math.unm.edu)). We are looking forward to hearing from you.

- 1) Prove the law of cosines: in a triangle with sides of lengths  $a$ ,  $b$  and  $c$  we have

$$c^2 = a^2 + b^2 - 2ab \cos \theta,$$

where  $\theta$  is the angle between the sides  $a$  and  $b$ . This is a generalization of the Pythagorean theorem as when  $\theta = \pi/2$  (i.e.  $90^\circ$ ) we obtain  $c^2 = a^2 + b^2$ .

- 2) Prove that  $3^n + 1$  is not divisible by  $2^n$  for any natural number  $n > 1$ .
- 3) Show that if we split the natural numbers  $1, 2, 3, 4, \dots$  in two disjoint subsets  $A$  and  $B$ , then one of the sets  $A$  or  $B$  contains the difference of two of its elements.
- 4) Let  $\theta$  be a fixed angle,  $0 < \theta < 180^\circ$ , and  $c > 0$  a fixed number. Inside the angle  $\theta$  find the location of all points whose sum of distances to the two arms of the angle is the given number  $c$ .
- 5) (a) Show that if  $a$  is any positive integer, then 14 divides  $a^7 - a$ .  
 (b) Prove that there are no positive integers  $a$  and  $b$  such that

$$a^7 + b^7 + 2 = (a - 1)^7 + (b + 4)^7.$$

- 6) Let  $a$  and  $b$  be two relatively prime positive integers.
- (a) Suppose  $1 < a < b$ . For each integer  $k$ ,  $0 \leq k \leq a - 1$ , let  $r_k$  be the remainder of the division of  $kb$  by  $a$ . In this way we obtain  $a$  non-negative integers  $r_k$ ,  $0 \leq r_k \leq a - 1$ . Show that the set  $\{r_k : 0 \leq k \leq a - 1\}$  is exactly the set  $\{0, 1, 2, \dots, a - 1\}$ , i.e., all possible remainders of divisions by  $a$ .
- (b) Show that for every integer  $n \geq (a - 1)(b - 1)$  there are non-negative integers  $x$  and  $y$  for which  $n = ax + by$ .
- (c) Find the smallest number  $n_o$  so that for any integer  $n$ ,  $n \geq n_o$ , there are non-negative integers  $x$  and  $y$  for which  $n = ax + by$ .